

# Duality between $\text{Chow}^2\mathbb{P}^4$ and the Double Quintic Symmetroids

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ABSTRACT. Let  $\mathcal{X}$  be the second symmetric product of  $\mathbb{P}^4$  and  $\mathcal{Y}$  the double cover of the symmetric determinantal quintic hypersurface in  $\mathbb{P}^{14}$ . We obtain homological properties of  $\mathcal{X}$  and  $\mathcal{Y}$  which indicate the homological projective duality between (suitable noncommutative resolutions of)  $\mathcal{X}$  and  $\mathcal{Y}$ . Among other things, we construct good desingularizations  $\tilde{\mathcal{X}}$  and  $\tilde{\mathcal{Y}}$  of  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively, and also a dual Lefschetz collection in  $\mathcal{D}^b(\tilde{\mathcal{X}})$  and a Lefschetz collection in  $\mathcal{D}^b(\tilde{\mathcal{Y}})$ . These are expected to give respective (dual) Lefschetz decompositions of suitable noncommutative resolutions of  $\mathcal{D}^b(\mathcal{X})$  and  $\mathcal{D}^b(\mathcal{Y})$ . The desingularization  $\tilde{\mathcal{Y}}$  also contains interesting birational geometries of  $\mathcal{Y}$ .

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## 1. INTRODUCTION

In the previous work [HoTa1], we have encountered an interesting new geometry of Reye congruences in dimension three through our study of mirror symmetry of Calabi-Yau manifolds. Reye congruences in dimension two have been attracting attention for long in relation to geometries of Enriques surfaces [Co]. In dimension three, they define smooth Calabi-Yau threefolds. It was found in [HoTa1] that each of these Calabi-Yau threefolds is paired with another smooth Calabi-Yau threefold which arises naturally in the projective geometry of Reye congruences.

Let  $\mathcal{X} = \mathbb{S}^2\mathbb{P}(V)$  be the symmetric product of the projective space  $\mathbb{P}(V) \cong \mathbb{P}^n$ . In terms of the so-called Chow form, we can embed  $\mathcal{X}$  into the projective space  $\mathbb{P}(\mathbb{S}^2V)$  of symmetric  $(n+1) \times (n+1)$  matrices. Then  $\mathcal{X}$  is identified with the Chow variety of 0-cycles of length two, and may also be identified with the rank  $\leq 2$  locus of  $\mathbb{P}(\mathbb{S}^2V)$  in the natural stratification by the matrix rank. It is a well-known fact in classical projective geometry that this stratification is reversed to the corresponding one in the dual projective space  $\mathbb{P}(\mathbb{S}^2V^*)$ .

The Reye congruences are defined as general linear sections of  $\mathcal{X}$  by  $(n+1)$  linear forms on  $\mathbb{P}(\mathbb{S}^2V)$ . Since giving  $(n+1)$  linear forms is equivalent to fixing a linear subspace  $L \simeq \mathbb{C}^{n+1} \subset \mathbb{S}^2V^*$  in the vector space  $\mathbb{S}^2V^*$  dual to  $\mathbb{S}^2V$ , the corresponding Reye congruence may be written by  $X = \mathcal{X} \cap \mathbb{P}(L^\perp)$ . In this form, one may notice that the Reye congruence is in accord with the Mukai’s constructions [Mu] of Fano manifolds associated to homogeneous spaces. For example, in his classification of prime Fano threefolds, the Fano threefolds of genus 7, 8, 9, 10 are constructed by a similar linear sections of suitable homogeneous spaces. Furthermore, as an outcome of his construction, in the case of genus 7 for example, it was found that the intermediate Jacobian of the Fano threefold is isomorphic to the Jacobian of the curve which is obtained as the ‘orthogonal’ linear section of the projective dual of the homogeneous space.

Observing this similarity, we first considered in [HoTa1] the Hessian hypersurface in the dual projective space by  $H = \mathcal{H} \cap \mathbb{P}(L)$ , where  $\mathcal{H}$  represents the rank  $\leq n$  locus in  $\mathbb{P}(\mathbb{S}^2V^*)$ . Assume that  $n$  is even. While the Reye congruence  $X$  is a

smooth Calabi-Yau manifold,  $H$  is a Calabi-Yau variety which is singular along a codimension two subvariety. The new geometry found in [loc.cit.] is a double covering  $Y \rightarrow H$  branched along the singular locus of  $H$ . It was shown that  $Y$  is a smooth Calabi-Yau threefold when  $n = 4$ .

To clarify the previous construction, let us note that  $\mathcal{H}$  is the determinantal hypersurface of degree  $n + 1$ . We call this as *symmetroid* in this paper. When  $n$  is even, we will define the double covering  $\mathcal{Y} \rightarrow \mathcal{H}$  branched along the rank  $\leq (n - 1)$  locus of  $\mathcal{H}$  (Proposition 4.2.2). We call this covering as *double symmetroid*.  $\mathcal{Y}$  is singular for  $n \geq 4$  but still has nice properties in view of the minimal model program (Proposition 4.2.4). We denote by  $\mathcal{Y} \cap \mathbb{P}(L)$  the pull-back of  $\mathcal{H} \cap \mathbb{P}(L)$  under  $\mathcal{Y} \rightarrow \mathcal{H}$ . We show that  $Y = \mathcal{Y} \cap \mathbb{P}(L)$  is a Calabi-Yau variety in general (Proposition 4.3.1), and is smooth when  $n = 4$ . Moreover, at the end of Subsection 6.9, we show that it coincides with the double covering Calabi-Yau threefolds previously found when  $n = 4$ .

The recent proposal in [Ku2, Ku4], called homological projective duality, describes the Mukai's construction in terms of the derived category of coherent sheaves and a suitable decomposition (Lefschetz decomposition) of it. More generally, for a singular variety, the proposal deals with the so-called noncommutative resolution which is a full subcategory of the derived category of a suitable resolution of the singularities. The classical examples of the Fano threefolds of genus 7, 8, 9, 10 due to Mukai, which are related to some nice homogeneous spaces, are described in this framework [Ku1]. Also the homological projective dual of the Grassmann variety  $G(2, 7)$  was shown to be the noncommutative resolution of its projective dual variety  $\text{Pf}(7)$ , the Pfaffian variety [Ku4]. Under this duality, two Calabi-Yau threefolds are obtained [BC, Ku4] as suitable linear sections in a similar way to the above. While we can observe many similarities between our case and the Grassmann-Pfaffian case, there are also many dis-similarities between the two. One important difference we should note is that  $\mathcal{X}$  as well as  $\mathcal{Y}$  are not homogeneous spaces although they admit natural quasi-homogeneous  $\text{SL}(n)$  actions.

In this article, we make a first step toward formulating the new geometry appeared in the previous work within the framework of the homological projective duality.

The homological projective duality, if applies to our case, provides a systematic way to describe the derived categories of the (noncommutative resolutions of) linear sections of  $\mathcal{X}$  and  $\mathcal{Y}$ . Showing the homological projective duality in general consists of two major steps of finding suitable categorical/noncommutative resolutions and making suitable (dual) Lefschetz decompositions of them. The categorical/noncommutative resolutions of  $\mathcal{X}$  and  $\mathcal{Y}$ , respectively, should be identified inside the derived categories  $\mathcal{D}^b(\tilde{\mathcal{X}})$  and  $\mathcal{D}^b(\tilde{\mathcal{Y}})$  as full subcategories by finding suitable resolutions  $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$  and  $\tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ .

A natural resolution of  $\mathcal{X}$  is given by the Hilbert-Chow morphism  $\tilde{\mathcal{X}} = \text{Hilb}^2 \mathbb{P}^n \rightarrow \mathcal{X}$  (Proposition 3.3.1). In this case, it should be rather easy to find the noncommutative resolution of  $\mathcal{X}$  in the derived category  $\mathcal{D}^b(\tilde{\mathcal{X}})$  based on the theory of [Ku3] (see [HoTa4]). In contrast to this, the singularity of  $\mathcal{Y}$  turns out to be more complicated (see the beginning of Section 7 for a brief summary). Because of this complication, the theorem [Ku3, Theorem 4.3] does not apply to this for example,

and having the noncommutative resolution seems to be a more difficult task although we will find a nice resolution  $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  (**Proposition 6.6.1**) based on the birational geometry of  $\mathcal{Y}$ . In this paper, as a strong indication for the homological projective duality between  $\mathcal{Y}$  and  $\mathcal{X}$ , we find a Lefschetz collection which generates a full subcategory of  $\mathcal{D}^b(\widetilde{\mathcal{Y}})$  (**Theorem 8.1.1**) and also the corresponding dual Lefschetz collection in  $\mathcal{D}^b(\widetilde{\mathcal{X}})$  (**Theorem 3.4.4**). These two theorems are main results of this paper. We expect that these (dual) Lefschetz collections are actually the (dual) Lefschetz decompositions of the noncommutative resolutions of  $\mathcal{Y}$  and  $\mathcal{X}$ .

The construction of this paper is as follows: In Section 3, we summarize some basic properties of the Chow variety  $\mathcal{X}$ , and construct the Hilbert-Chow morphism  $\widetilde{\mathcal{X}} \rightarrow \mathcal{X}$ . Using these, we construct the dual Lefschetz collection in  $\mathcal{D}^b(\widetilde{\mathcal{X}})$ . Also some basic properties of the Calabi-Yau manifolds of Reye congruences are summarized. In Section 4, we introduce the determinantal hypersurface (symmetroid)  $\mathcal{H}$ , and using the geometry of singular quadric parametrized by  $\mathcal{H}$ , we define its double cover, i.e., the double symmetroid  $\mathcal{Y}$ . We define a Calabi-Yau variety  $Y$  in  $\mathcal{Y}$ , and for  $n = 4$ , we determine topological invariants of  $Y$  from geometries of  $\mathcal{Y}$  (**Proposition 4.3.4**). In Section 5, we elaborate the birational geometry of  $\mathcal{Y}$  by relating it with the Hilbert scheme of conics on Grassmann  $G(3, V)$  (Proposition 5.3.1). There, we classify such conics and summarize basic properties of them. In Section 6, we further develop our study of the birational geometry of  $\mathcal{Y}$ . In particular the Grassmann bundle  $G(3, T(-1)^{\wedge 2})$  over  $\mathbb{P}(V)$  will be introduced, and the so-called two ray game (Sarkisov link) of this Grassmann bundle is studied in detail to obtain the announced desingularization  $\widetilde{\mathcal{Y}}$  of  $\mathcal{Y}$  (see Fig.2 and the diagram (6.20) for summary). The morphism  $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  turns out to be a divisorial contraction which is negative with respect to the canonical divisor (Propositions 6.9.1 and 6.9.2). Through the construction of  $\widetilde{\mathcal{Y}}$  from  $G(3, T(-1)^{\wedge 2})$ , we find three locally free sheaves  $\widetilde{\mathcal{S}}_L$ ,  $\widetilde{\mathcal{Q}}$ , and  $\widetilde{\mathcal{T}}$  on  $\widetilde{\mathcal{Y}}$  (Definitions 6.7.1 and 6.7.3), which generate a Lefschetz collection in  $\mathcal{D}^b(\widetilde{\mathcal{Y}})$ . It should be emphasized here that our viewpoint of  $\mathcal{Y}$  as a birational model of the Hilbert scheme of conics on  $G(3, V)$  enables us to associate moduli theoretic meaning for each of the birational models of  $G(3, T(-1)^{\wedge 2})$  (see Subsection 6.8). In Section 7, we further study the exceptional divisor  $F_{\widetilde{\mathcal{Y}}}$  of divisorial contraction  $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  by describing geometries of double lines on  $G(3, V)$ , i.e., conics of rank one. We determine the global structure of  $F_{\widetilde{\mathcal{Y}}}$  in Proposition 7.2.5 and special fibers of  $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  in Proposition 7.3.1. This section is necessary for our cohomology calculations in the subsequent section. In Section 8, we find a Lefschetz collection in  $\mathcal{D}^b(\widetilde{\mathcal{Y}})$  and observe a certain duality between the quiver diagrams associated to the Lefschetz collection in  $\mathcal{D}^b(\widetilde{\mathcal{Y}})$  and the dual Lefschetz collection in  $\mathcal{D}^b(\widetilde{\mathcal{X}})$  respectively ((3.9) and (8.1)).

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### Glossary of notation.

Throughout the paper, we work over  $\mathbb{C}$ , the complex number field.

We often identify Cartier divisors with the corresponding invertible sheaves.

$\mathbb{P}(W)$ : the projectivization of a vector space  $W$ .

$\mathcal{O}_X(1) := \mathcal{O}_{\mathbb{P}(W)}(1)|_X$  if a variety  $X$  is naturally embedded in  $\mathbb{P}(W)$ .

$\mathbb{P}(\mathcal{E})$ : the projectivization of a locally free sheaf  $\mathcal{E}$  on a variety  $X$ .

$H_{\mathbb{P}(\mathcal{E})}$ : the tautological divisor of  $\mathbb{P}(\mathcal{E})$ .

$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ : the tautological invertible sheaf of  $\mathbb{P}(\mathcal{E})$ .

$G(r, \mathcal{E})$ : the Grassmann bundle which parametrizes  $(r-1)$ -dimensional linear subspaces in fibers of  $\mathbb{P}(\mathcal{E})$ .

$V$ : a (fixed)  $n+1$  dimensional complex vector space.  $\mathbb{P}^n := \mathbb{P}(V)$ .

$V_i$ : an  $i$ -dimensional vector subspace of  $V$ .

$\mathcal{X} = \text{Chow}^2 \mathbb{P}(V)$ : the Chow variety of 0-cycles of length two in  $\mathbb{P}(V)$ .

$\check{\mathcal{X}} = \text{Hilb}^2 \mathbb{P}(V)$ : the Hilbert scheme of 0-dimensional subschemes of length two in  $\mathbb{P}(V)$ .

$L_{\check{\mathcal{X}}}$ : the pull-back of  $\mathcal{O}_{G(2,V)}(1)$  by the morphism  $\check{\mathcal{X}} \rightarrow G(2, V)$ .

$H_{\check{\mathcal{X}}}$ : the pull-back of  $\mathcal{O}_{\mathcal{X}}(1)$  by the morphism  $\check{\mathcal{X}} \rightarrow \mathcal{X}$ .

$\mathcal{H}$ : the symmetroid of degree  $n+1$  in  $\mathbb{P}(S^2 V^*)$ .

$\mathcal{Y}$ : the double symmetroid, i.e., the double cover of  $\mathcal{H}$  ramified along the locus of quadrics in  $\mathbb{P}(V)$  of rank less than or equal to  $n-1$  for an even  $n$ .

$P$ : a general  $n$ -plane in  $\mathbb{P}(S^2 V^*)$ .

$X := \mathcal{X} \cap P^\perp$ , which is a Calabi-Yau manifold of dimension  $n-1$  if  $n$  is even.

$H := \mathcal{H} \cap P$ , the symmetric determinantal hypersurface with trivial canonical class.

$Y$ : the pull-back of  $H$  by  $\mathcal{Y} \rightarrow \mathcal{H}$ , which is a Calabi-Yau variety of dimension  $n-1$  for an even  $n$ .

$F(a, b, V) := \{(V_a, V_b) \mid V_a \subset V_b \subset V\}$  (the flag variety).

The universal exact sequences on the Grassmannians for  $V$  ( $n=4$ ):

$$0 \rightarrow \mathcal{F}^* \rightarrow V \otimes \mathcal{O}_{G(2,V)} \rightarrow \mathcal{G} \rightarrow 0,$$

$$0 \rightarrow \mathcal{U}^* \rightarrow V \otimes \mathcal{O}_{G(3,V)} \rightarrow \mathcal{W} \rightarrow 0,$$

$\mathcal{F}$ : the dual of the universal subbundle of rank two on  $G(2, V)$ ,

$\mathcal{G}$ : the universal quotient bundle of rank three on  $G(2, V)$ ,

$\mathcal{U}$ : the dual of the universal subbundle of rank three on  $G(3, V)$ ,

$\mathcal{W}$ : the universal quotient bundle of rank two on  $G(3, V)$ .

We also use the following notation which simplifies lengthy formulas:

$$\Omega(1) := \Omega_{\mathbb{P}(V)}(1).$$

$$\Omega(1)^{\wedge i} := \wedge^i(\Omega_{\mathbb{P}(V)}(1)) \text{ for } i \geq 2.$$

$$T(-1) := T_{\mathbb{P}(V)}(-1).$$

$$T(-1)^{\wedge i} := \wedge^i(T(-1)) \text{ for } i \geq 2.$$

$$\mathcal{O}(i) := \mathcal{O}_{\mathbb{P}(V)}(i) \text{ for } i \in \mathbb{Z}.$$

## 2. PRELIMINARIES

For the computations of cohomology groups which appear in this paper, we use the Bott theorem about the cohomology groups of Grassmann bundles below extensively.

For a locally free sheaf  $\mathcal{E}$  of rank  $r$  on a variety and a nonincreasing sequence  $\beta = (\beta_1, \beta_2, \dots, \beta_r)$  of integers, we denote by  $\Sigma^\beta \mathcal{E}$  the associated locally free sheaf with the Schur functor  $\Sigma^\beta$ .

**Theorem 2.0.1. (Bott theorem)** *Let  $\pi: G(r, \mathcal{A}) \rightarrow X$  be a Grassmann bundle for a locally free sheaf  $\mathcal{A}$  on a variety  $X$  of rank  $n$  and  $0 \rightarrow \mathcal{S}^* \rightarrow \mathcal{A} \rightarrow \mathcal{Q} \rightarrow 0$  the universal exact sequence. For  $\beta := (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}^r$  ( $\alpha_1 \geq \dots \geq \alpha_r$ ) and  $\gamma := (\alpha_{r+1}, \dots, \alpha_n) \in \mathbb{Z}^{n-r}$  ( $\alpha_{r+1} \geq \dots \geq \alpha_n$ ), we set  $\alpha := (\beta, \gamma)$  and  $\mathcal{V}(\alpha) := \Sigma^\beta \mathcal{S} \otimes \Sigma^\gamma \mathcal{Q}^*$ . Finally, let  $\rho := (n, \dots, 1)$ , and, for an element  $\sigma$  of the  $n$ -th symmetric group  $\mathfrak{S}_n$ , we set  $\sigma^\bullet(\alpha) := \sigma(\alpha + \rho) - \rho$ .*

- (1) *If  $\sigma(\alpha + \rho)$  contains two equal integers, then  $R^i \pi_* \mathcal{V}(\alpha) = 0$  for any  $i \geq 0$ .*
- (2) *If there exists an element  $\sigma \in \mathfrak{S}_n$  such that  $\sigma(\alpha + \rho)$  is strictly decreasing, then  $R^i \pi_* \mathcal{V}(\alpha) = 0$  for any  $i \geq 0$  except  $R^{l(\sigma)} \pi_* \mathcal{V}(\alpha) = \Sigma^{\sigma^\bullet(\alpha)} \mathcal{A}^*$ , where  $l(\sigma)$  represents the length of  $\sigma \in \mathfrak{S}_n$ .*

*Proof.* See [Bo], [D], or [W, (4.19) Corollary]. □

In this paper, we adopt the following definition of Calabi-Yau variety and also Calabi-Yau manifold.

**Definition 2.0.2.** We say a normal projective variety  $X$  a *Calabi-Yau variety* if  $X$  has only Gorenstein canonical singularities, the canonical bundle of  $X$  is trivial, and  $h^i(\mathcal{O}_X) = 0$  for  $0 < i < \dim X$ . If  $X$  is smooth, then  $X$  is called a *Calabi-Yau manifold*. A smooth Calabi-Yau threefold is abbreviated as a Calabi-Yau threefold. □

3. GEOMETRIES OF  $\text{Chow}^2 \mathbb{P}(V)$ 3.1.  $\text{Chow}^2 \mathbb{P}(V)$  and quadrics in  $\mathbb{P}(V)$ .

Let

$$\mathcal{X} := \text{Chow}^2 \mathbb{P}(V)$$

be the Chow variety of 0-cycles in  $\mathbb{P}(V)$  of length 2.  $\mathcal{X}$  is embedded in the projective space  $\mathbb{P}(\mathcal{S}^2 V)$  for the symmetric product  $\mathcal{S}^2 V := \text{Sym}^2 V$ . When we denote the homogeneous coordinate of  $\mathbb{P}(\mathcal{S}^2 V)$  by  $w_{ij} = w_{ji}$  ( $1 \leq i, j \leq n+1$ ), the embedding is given by the so-called *Chow form*;

$$(3.1) \quad w_{ii} = x_i y_i, \quad w_{ij} := x_i y_j + x_j y_i \quad (i \neq j),$$

where  $x_i, y_i$  ( $1 \leq i \leq n+1$ ) are coordinates of  $\mathbb{P}(V)$  (cf. [GKZ, Theorem 2.2]). We may view this as a morphism  $\mathbb{P}(V) \times \mathbb{P}(V) \rightarrow \mathbb{P}(\mathcal{S}^2 V)$ . Then, since the  $w_{ij}$  generate all the polynomials symmetric under  $x_i \leftrightarrow y_i$ , we naturally have an isomorphism  $\mathcal{X} \simeq \mathcal{S}^2 \mathbb{P}(V)$ . Then we may identify  $\mathbb{P}(\mathcal{S}^2 V)$  with the dual of the space of the symmetric  $(1, 1)$ -divisors on  $\mathbb{P}(V) \times \mathbb{P}(V)$ . Moreover,  $\mathcal{S}^2 \mathbb{P}(V)$  is defined by all the

$3 \times 3$  minors of the generic  $(n+1) \times (n+1)$  symmetric matrix  $(w_{ij})$ , namely,  $S^2\mathbb{P}(V)$  is the locus of symmetric matrices of rank less than or equal to two.

On the other hand, a quadric in  $\mathbb{P}(V)$  is represented as  ${}^t\mathbf{x}A\mathbf{x} = 0$  with  ${}^t\mathbf{x} = (x_1, \dots, x_{n+1})$  and an  $(n+1) \times (n+1)$  symmetric matrix  $A$ . We denote by  $Q_A$  the quadric defined by a symmetric matrix  $A$ , and by  $q_A(\mathbf{x})$  the quadratic form  ${}^t\mathbf{x}A\mathbf{x}$ . The projectivization of the vector space of all the  $(n+1) \times (n+1)$  symmetric matrices may be identified with  $\mathbb{P}(S^2V^*)$ , which is dual to  $\mathbb{P}(S^2V)$ . More explicitly, we can write the dual pairing by

$$A \cdot w = \sum_{1 \leq i \leq j \leq n+1} a_{ij} w_{ij},$$

where  $w = (w_{ij}) \in \mathbb{P}(S^2V)$ . By (3.1), we have the equality

$$(3.2) \quad A \cdot w_{\mathbf{x}\mathbf{y}} = {}^t\mathbf{x}A\mathbf{y},$$

where  $w_{\mathbf{x}\mathbf{y}}$  is the image in  $\mathbb{P}(S^2V)$  of  $(\mathbf{x}, \mathbf{y}) \in \mathbb{P}(V) \times \mathbb{P}(V)$ .

### 3.2. Projective duality of $\text{Sec}^i \mathcal{X}_0$ .

When we restrict the morphism  $\mathbb{P}(V) \times \mathbb{P}(V) \rightarrow \mathbb{P}(S^2V)$  obtained as above to the diagonal, we obtain the second Veronese morphism of  $\mathbb{P}(V)$ . We denote by  $\mathcal{X}_0 = v_2(\mathbb{P}(V))$  its image. Then it is easy to see that  $S^2\mathbb{P}(V)$  is defined by all the  $2 \times 2$  minors of the generic  $(n+1) \times (n+1)$  symmetric matrix  $(w_{ij})$ , namely,  $\mathcal{X}_0$  is the locus of symmetric matrices of rank one. By the characterizations of  $\mathcal{X}$  and  $\mathcal{X}_0$  with rank condition as above, we see that  $\mathcal{X}$  is the secant variety of  $\mathcal{X}_0$ , namely,

$$\mathcal{X} = \overline{\cup \{ \langle p, q \rangle \mid p, q \in \mathcal{X}_0, p \neq q \}},$$

where  $\langle p, q \rangle$  is the line through  $p$  and  $q$ .

The second Veronese variety  $\mathcal{X}_0$  is one of the Scorza varieties classified by Zak (see [Z], [Ch]). Associated to  $\mathcal{X}_0$ , we naturally have the tower of higher secant varieties. Recall that, for projective varieties  $X, Y \subset \mathbb{P}^N$ , in general, their *join* is defined as

$$J(X, Y) := \overline{\cup \{ \langle p, q \rangle \mid p \in X, q \in Y, p \neq q \}}.$$

The higher secant varieties are defined inductively by  $\text{Sec}^m X := J(\text{Sec}^{m-1} X, X)$  with  $\text{Sec}^0 X := X$ . Note that  $\text{Sec}^1 X$  is nothing but the secant variety of  $X$ . Since  $\mathcal{X}_0$  is the locus of symmetric matrices of rank one, one may identify  $\text{Sec}^i \mathcal{X}_0$  with the locus of symmetric matrices of rank  $\leq i+1$  since a general point of  $\text{Sec}^i \mathcal{X}_0$  corresponds to a sum of  $(i+1)$  matrices of rank one. In particular, it holds that  $\text{Sec}^n \mathcal{X}_0 = \mathbb{P}(S^2V)$  and  $\text{Sec}^{n-1} \mathcal{X}_0$  is the hypersurface of degree  $n+1$  defined by the determinant of the generic symmetric matrix  $(w_{ij})$ . In summary, we have the tower of the secant varieties:

$$(3.3) \quad \begin{array}{ccccccc} \emptyset & \subset & \mathcal{X}_0 & \subset & \text{Sec}^1 \mathcal{X}_0 & \subset & \text{Sec}^2 \mathcal{X}_0 & \subset & \dots & \subset & \text{Sec}^{n-1} \mathcal{X}_0 & \subset & \text{Sec}^n \mathcal{X}_0. \\ & & \parallel & & \parallel & & & & & & \parallel & & \\ & & v_2(\mathbb{P}(V)) & & \mathcal{X} & & & & & & \mathbb{P}(S^2V) & & \end{array}$$

It is known that  $\dim \text{Sec}^i \mathcal{X}_0 = (i+1)n - \frac{i(i-1)}{2}$ . This tower gives the orbit decomposition of the action of  $\text{SL}_{n+1}$  on  $\mathbb{P}(S^2V)$ . Precisely,  $\text{Sec}^i \mathcal{X}_0 \setminus \text{Sec}^{i-1} \mathcal{X}_0$  is an  $\text{SL}_{n+1}$ -orbit for any  $i$  since any two symmetric matrices of the same rank are transformed to each other by  $\text{SL}_{n+1}$ . It is known that  $\text{Sec}^{i+1} \mathcal{X}_0$  is the singular locus of  $\text{Sec}^i \mathcal{X}_0$ . In particular,  $\mathcal{X}_0 = \text{Sing } \mathcal{X}$ .



In the dual projective space  $\mathbb{P}(\mathbb{S}^2 V^*)$  to  $\mathbb{P}(\mathbb{S}^2 V)$ , we consider the dual varieties  $(\text{Sec}^i \mathcal{X}_0)^*$  of  $\text{Sec}^i \mathcal{X}_0$ , namely,  $(\text{Sec}^i \mathcal{X}_0)^*$  is the closure of the locus of the hyperplanes of  $\mathbb{P}(\mathbb{S}^2 V)$  tangent to  $\text{Sec}^i \mathcal{X}_0$  at smooth points. Verifying the tangent space of  $\text{Sec}^i \mathcal{X}_0$  at a general point, we see that  $(\text{Sec}^i \mathcal{X}_0)^*$  is the locus of symmetric matrices  $A$  of rank  $\leq n - i$ . In summary, we have the dual tower to (3.3):

$$(3.4) \quad \begin{array}{ccccccc} \mathbb{P}(\mathbb{S}^2 V^*) & \supset & (\mathcal{X}_0)^* & \supset & (\text{Sec}^1 \mathcal{X}_0)^* & \supset & (\text{Sec}^2 \mathcal{X}_0)^* \supset \cdots \supset (\text{Sec}^{n-1} \mathcal{X}_0)^* \supset \emptyset. \\ & & \parallel & & \parallel & & \\ & & \mathcal{H} & & \mathcal{X}^* & & \end{array}$$

Throughout this paper, we set

$$\mathcal{H} := (\mathcal{X}_0)^*.$$

$\mathcal{H}$  is the symmetric determinantal hypersurface in  $\mathbb{P}(\mathbb{S}^2 V^*)$ , which is called *the symmetroid*.

### 3.3. $\tilde{\mathcal{X}} = \text{Hilb}^2 \mathbb{P}(V)$ and $\mathbf{G}(2, V)$ .

By the construction of the double cover  $\mathbb{P}(V) \times \mathbb{P}(V) \rightarrow \mathcal{X}$ , we see that  $\mathcal{X}$  has quotient singularities of type  $\frac{1}{2}(1^n)$  along the singular locus  $\mathcal{X}_0 = v_2(\mathbb{P}(V))$ . In particular,  $\mathcal{X}$  is Gorenstein if and only if  $n$  is even.

Hereafter in this subsection, we assume that  $n$  is even. We set

$$\tilde{\mathcal{X}} := \text{Hilb}^2 \mathbb{P}(V),$$

which is the Hilbert scheme of 0-dimensional subscheme of  $\mathbb{P}(V)$  of length 2 (we simply call it the Hilbert scheme of two points on  $\mathbb{P}(V)$ ). Recall that the Hilbert-Chow morphism  $f: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  is the blow-up of  $\mathcal{X}$  along the singular locus  $\mathcal{X}_0$ . Therefore we have

$$(3.5) \quad K_{\tilde{\mathcal{X}}} = -(n+1)H_{\tilde{\mathcal{X}}} + \frac{n-2}{2}E_f.$$

Since the 0-dimensional subscheme on  $\mathbb{P}(V)$  of length two determines a line on  $\mathbb{P}(V)$ , we have a natural morphism  $g: \tilde{\mathcal{X}} \rightarrow \mathbf{G}(2, V)$ :

$$\begin{array}{ccc} & \tilde{\mathcal{X}} & \\ f \swarrow & & \searrow g \\ \mathcal{X} & & \mathbf{G}(2, V). \end{array}$$

Let  $\mathcal{F}$  be the dual of the universal subbundle of rank two on  $\mathbf{G}(2, V)$ .

**Proposition 3.3.1.** *The Hilbert scheme  $\tilde{\mathcal{X}}$  of two points on  $\mathbb{P}(V)$  is isomorphic to  $\mathbb{P}(\mathbb{S}^2 \mathcal{F}^*)$ . In particular,  $f: \tilde{\mathcal{X}} \rightarrow \mathcal{X}$  is a desingularization of  $\mathcal{X}$ .*

*Proof.* By definition,  $\mathbb{P}(\mathcal{F}^*) \rightarrow \mathbf{G}(2, V)$  is the universal family of lines on  $\mathbb{P}(V)$ . The fiber of  $g$  over a point  $[l] \in \mathbf{G}(2, V)$  is identified with  $\text{Hilb}^2 l = \mathbb{S}^2 l$ . Hence the claim follows.  $\square$

We set

$$H_{\tilde{\mathcal{X}}} = f^* \mathcal{O}_{\mathcal{X}}(1) \text{ and } L_{\tilde{\mathcal{X}}} = g^* \mathcal{O}_{\mathbf{G}(2, V)}(1).$$

By Proposition 3.3.1, we see that  $f$  follows from the natural morphism  $\mathbb{P}(\mathbb{S}^2 \mathcal{F}^*) \rightarrow \mathbb{P}(\mathbb{S}^2 V)$ . Hence  $H_{\tilde{\mathcal{X}}}$  is the tautological divisor for  $\mathbb{P}(\mathbb{S}^2 \mathcal{F}^*)$ .



We denote by  $E_f$  the  $f$ -exceptional divisor. We see that  $E_f \simeq \mathbb{P}(\mathcal{F}^*)$ , namely,  $E_f$  is isomorphic to the total space of the universal family of lines on  $\mathbb{P}(V)$  since  $E_f \subset \text{Hilb}^2 \mathbb{P}(V)$  parameterizes pairs of points  $x \in \mathbb{P}(V)$  and lines  $l$  through  $x$ .

Moreover, from the relative Euler sequence  $0 \rightarrow \mathcal{O}_{\check{\mathcal{X}}} \rightarrow g^*(S^2 \mathcal{F}^*) \otimes \mathcal{O}_{\check{\mathcal{X}}}(1) \rightarrow T_{\check{\mathcal{X}}/G(2,V)} \rightarrow 0$ , we have

$$(3.6) \quad K_{\check{\mathcal{X}}} = -3H_{\check{\mathcal{X}}} - (n-2)L_{\check{\mathcal{X}}}.$$

By (3.5) and (3.6), we have

$$(3.7) \quad \frac{n-2}{2}E_f \sim (n-2)(H_{\check{\mathcal{X}}} - L_{\check{\mathcal{X}}}).$$

### 3.4. Constructing a dual Lefschetz collection in $\mathcal{D}^b(\check{\mathcal{X}})$ .

We recall some basic definitions from the theory of triangulated categories (cf. [BO]).

**Definition 3.4.1.** An object  $\mathcal{E}$  in a triangulated category  $\mathcal{D}$  is called an *exceptional object* if  $\text{Hom}(\mathcal{E}, \mathcal{E}) \simeq \mathbb{C}$  and  $\text{Hom}^\bullet(\mathcal{E}, \mathcal{E}) \simeq 0$  for  $\bullet \neq 0$ .  $\square$

**Definition 3.4.2.** A sequence  $\mathcal{D}_1, \dots, \mathcal{D}_m$  of admissible triangulated subcategories in a triangulated category  $\mathcal{D}$  is called a *semiorthogonal collection* if  $\text{Hom}_{\mathcal{D}}(\mathcal{D}_i, \mathcal{D}_j) = 0$  for any  $i > j$ . Moreover, if  $\mathcal{D}_1, \dots, \mathcal{D}_m$  generates  $\mathcal{D}$ , then it is called a *semiorthogonal decomposition*.

A semiorthogonal collection of exceptional objects  $\mathcal{E}_1, \dots, \mathcal{E}_n$  is called an *exceptional collection*. Moreover, if  $\text{Ext}^\bullet(\mathcal{E}_i, \mathcal{E}_j) \simeq 0$  holds for any  $i, j$  and any  $\bullet > 0$ , then it is called an *strongly exceptional collection*.  $\square$

Hereafter, in this article, we restrict our attention to the cases of the derived categories of bounded complexes of coherent sheaves on a variety. In such cases, a special type of semiorthogonal collection plays an important role (cf. [Ku2, Ku4]).

**Definition 3.4.3.** For a variety  $X$ , a *Lefschetz collection* of  $\mathcal{D}^b(X)$  is a semiorthogonal collection of the following form:

$$\mathcal{D}_0, \mathcal{D}_1(1), \dots, \mathcal{D}_{m-1}(m-1),$$

where it holds that  $0 \subset \mathcal{D}_{m-1} \subset \mathcal{D}_{m-2} \subset \dots \subset \mathcal{D}_0 \subset \mathcal{D}^b(X)$  and  $(k)$  means the twist by  $L^{\otimes k}$  with a fixed invertible sheaf  $L$ . Moreover, if  $\mathcal{D}_0, \mathcal{D}_1(1), \dots, \mathcal{D}_{m-1}(m-1)$  generate  $\mathcal{D}^b(X)$ , then it is called a *Lefschetz decomposition*.

Similarly, a *dual Lefschetz collection* of  $\mathcal{D}^b(X)$  is a semiorthogonal collection of the following form:

$$\mathcal{D}_{m-1}(-(m-1)), \mathcal{D}_{m-2}(-(m-2)), \dots, \mathcal{D}_0,$$

where it holds that  $0 \subset \mathcal{D}_{m-1} \subset \mathcal{D}_{m-2} \subset \dots \subset \mathcal{D}_0 \subset \mathcal{D}^b(X)$ . Moreover, if  $\mathcal{D}_{m-1}(-(m-1)), \mathcal{D}_{m-2}(-(m-2)), \dots, \mathcal{D}_0$  generate  $\mathcal{D}^b(X)$ , then it is called a *dual Lefschetz decomposition*.  $\square$

Now, based on the geometry of the projective bundle  $\check{\mathcal{X}} = \mathbb{P}(S^2 \mathcal{F}^*)$  over  $G(2, V)$ , we construct a dual Lefschetz collection in  $\mathcal{D}^b(\check{\mathcal{X}})$  by restricting our attention to the case  $n = 4$ . Other cases of  $n > 4$  should be done in a similar way, but we confine ourselves to this case to avoid possible complications.

We may naturally conceive the sheaves  $\mathcal{O}_{\check{\mathcal{X}}}$  and  $g^*\mathcal{F}$  as the objects in the (dual) Lefschetz collection. Recall by Proposition 3.3.1, we have  $\check{\mathcal{X}} = \mathbb{P}(\mathbb{S}^2\mathcal{F}^*)$  and associated Euler sequence  $0 \rightarrow \mathcal{O}_{\mathbb{P}(\mathbb{S}^2\mathcal{F}^*)}(-1) \rightarrow g^*\mathbb{S}^2\mathcal{F}^* \rightarrow T_{\mathbb{P}(\mathbb{S}^2\mathcal{F}^*)/G(2,V)}(-1) \rightarrow 0$ . Twisting this by  $2L_{\check{\mathcal{X}}}$  we obtain an injection

$$(3.8) \quad \varphi : \mathcal{O}_{\check{\mathcal{X}}}(-H_{\check{\mathcal{X}}} + 2L_{\check{\mathcal{X}}}) \rightarrow (g^*\mathbb{S}^2\mathcal{F}^*)(2L_{\check{\mathcal{X}}}) \simeq g^*\mathbb{S}^2\mathcal{F},$$

where we use  $\mathcal{F}^* \otimes \mathcal{O}_{G(2,V)}(1) = \Sigma^{(0,-1)}\mathcal{F} \otimes \Sigma^{(1,1)}\mathcal{F} \simeq \mathcal{F}$ . The cokernel of this injection, which is also  $T_{\mathbb{P}(\mathbb{S}^2\mathcal{F}^*)/G(2,V)}(-H_{\check{\mathcal{X}}} + 2L_{\check{\mathcal{X}}})$ , plays a role in the following theorem:

**Theorem 3.4.4.** (1) *Let*

$$(\mathcal{F}_3, \mathcal{F}_2, \mathcal{F}_{1a}, \mathcal{F}_{1b}) = (\mathcal{O}_{\check{\mathcal{X}}}, g^*\mathcal{F}, \text{Coker } \varphi, \mathcal{O}_{\check{\mathcal{X}}}(L_{\check{\mathcal{X}}}))$$

*be an ordered collection of sheaves on  $\check{\mathcal{X}}$ . Then  $(\mathcal{K}_i)_{1 \leq i \leq 4} := (\mathcal{F}_{1b}^*, \mathcal{F}_{1a}^*, \mathcal{F}_2^*, \mathcal{F}_3^*)$  is a strongly exceptional collection of  $\mathcal{D}^b(\check{\mathcal{X}})$ , namely satisfies*

$$H^\bullet(\mathcal{K}_i^* \otimes \mathcal{K}_j) = 0 \text{ for } 1 \leq i, j \leq 4 \text{ and } \bullet > 0$$

*and  $H^0(\mathcal{K}_i^* \otimes \mathcal{K}_j) = 0$  ( $i > j$ ),  $H^0(\mathcal{K}_i^* \otimes \mathcal{K}_i) \simeq \mathbb{C}$  ( $1 \leq i \leq 4$ ).*

(2) *For  $i < j$ ,  $H^0(\mathcal{K}_i^* \otimes \mathcal{K}_j)$  are given by*

$$H^0(\mathcal{F}_3^* \otimes \mathcal{F}_2) \simeq V^*, H^0(\mathcal{F}_3^* \otimes \mathcal{F}_{1a}) \simeq \mathbb{S}^2 V^*, H^0(\mathcal{F}_3^* \otimes \mathcal{F}_{1b}) \simeq \wedge^2 V^*,$$

$$H^0(\mathcal{F}_2^* \otimes \mathcal{F}_{1a}) \simeq V^*, H^0(\mathcal{F}_2^* \otimes \mathcal{F}_{1b}) \simeq V^*, H^0(\mathcal{F}_{1a}^* \otimes \mathcal{F}_{1b}) \simeq 0,$$

*which may be summarized in the following quiver diagram:*

$$(3.9) \quad \begin{array}{ccccc} & & \mathbb{S}^2 V^* & \xrightarrow{\quad} & \mathcal{F}_{1a} \\ & \nearrow & & \nearrow & \\ \mathcal{F}_3 & \xrightarrow{V^*} & \mathcal{F}_2 & \xrightarrow{V^*} & \\ & \searrow & & \searrow & \\ & & \wedge^2 V^* & \xrightarrow{\quad} & \mathcal{F}_{1b} \end{array}$$

(3) *Set  $\mathcal{D}_{\check{\mathcal{X}}} := \langle \mathcal{F}_{1b}^*, \mathcal{F}_{1a}^*, \mathcal{F}_2^*, \mathcal{F}_3^* \rangle \subset \mathcal{D}^b(\check{\mathcal{X}})$ . Then*

$$\mathcal{D}_{\check{\mathcal{X}}}(-4), \dots, \mathcal{D}_{\check{\mathcal{X}}}(-1), \mathcal{D}_{\check{\mathcal{X}}}$$

*is a dual Lefschetz collection, where  $(-t)$  represents the twist by the sheaf  $\mathcal{O}_{\check{\mathcal{X}}}(-tH_{\check{\mathcal{X}}})$ .*

We prepare the following lemma for our proof of the theorem.

**Lemma 3.4.5.** *Set  $\mathcal{C}_{ij} = \mathcal{K}_i^* \otimes \mathcal{K}_j$  ( $1 \leq i, j \leq 4$ ). For  $\mathcal{C} = \mathcal{C}_{ij}$ , it holds that*

$$H^\bullet(\check{\mathcal{X}}, \mathcal{C}(-t)) \simeq H^{8-\bullet}(\check{\mathcal{X}}, \mathcal{C}^*(t-5)) \text{ for any } t.$$

*Proof.* By the Serre duality and  $K_{\check{\mathcal{X}}} = -5H_{\check{\mathcal{X}}} + E_f$ , we have  $H^\bullet(\check{\mathcal{X}}, \mathcal{C}(-t)) \simeq H^{8-\bullet}(\check{\mathcal{X}}, \mathcal{C}^*((t-5)H_{\check{\mathcal{X}}} + E_f))$  for any  $t$ . By the exact sequence

$$0 \rightarrow \mathcal{C}^*((t-5)) \rightarrow \mathcal{C}^*((t-5)H_{\check{\mathcal{X}}} + E_f) \rightarrow \mathcal{C}^*((t-5)H_{\check{\mathcal{X}}} + E_f)|_{E_f} \rightarrow 0,$$

we have only to show that  $H^{8-\bullet}(E_f, \mathcal{C}^*((t-5)H_{\mathcal{X}} + E_f)|_{E_f}) = 0$ . Since  $E_f \rightarrow \mathcal{X}_0 \simeq \mathbb{P}^4$  is a  $\mathbb{P}^3$ -bundle, it suffices to show the vanishing of cohomology groups of the restriction of  $\mathcal{C}^*((t-5)H_{\mathcal{X}} + E_f)|_{E_f}$  to a fiber  $\Gamma$  of  $E_f \rightarrow \mathcal{X}_0$ . Note that  $\mathcal{O}_{\mathcal{X}}(E_f)|_{\Gamma} \simeq \mathcal{O}_{\mathbb{P}^3}(-2)$  and  $\mathcal{O}_{\mathcal{X}}(H_{\mathcal{X}})|_{\Gamma} \simeq \mathcal{O}_{\mathbb{P}^3}$ . As we note in the end of Subsection 3.3,  $E_f$  parameterizes pairs of a point  $x \in \mathbb{P}(V)$  and a line  $l$  through  $x$ . Therefore a fiber  $\Gamma \cong \mathbb{P}^3$  parameterizes lines through one fixed point. This implies that  $g^*\mathcal{F}|_{\Gamma} \simeq \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$ . Restricting the natural injection  $\mathcal{O}_{\mathcal{X}}(-H_{\mathcal{X}}) \rightarrow g^*\mathcal{S}^2\mathcal{F}^*$  to  $\Gamma$ , we have an injection

$$\mathcal{O}_{\mathbb{P}^3} \rightarrow \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2).$$

Therefore, by the definition of  $\mathcal{F}_{1a}$ , we have  $\mathcal{F}_{1a}|_{\Gamma} \simeq \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)$ . Consequently,  $\mathcal{C}^*((t-5)H_{\mathcal{X}} + E_f)|_{\Gamma}$  is a direct sum of  $\mathcal{O}_{\mathbb{P}^3}(-1)$ ,  $\mathcal{O}_{\mathbb{P}^3}(-2)$  and  $\mathcal{O}_{\mathbb{P}^3}(-3)$  for any  $\mathcal{C} = \mathcal{C}_{ij}$  and  $t$ , hence all of its cohomology groups vanish.  $\square$

*Proof of Theorem 3.4.4.* @ Note that, for the proof of (3), it suffices to verify

$$H^{\bullet}(\mathcal{K}_i^* \otimes \mathcal{K}_j(-t)) = 0 \quad (1 \leq i, j \leq 4, 1 \leq t \leq 4)$$

for  $\bullet > 0$ , since  $\mathcal{D}_{\mathcal{X}}$  is generated by  $\mathcal{K}_i$  ( $1 \leq i \leq 4$ ). Therefore, the claims (1),(2),(3) follow from explicit evaluations of the cohomology groups. Since the computations are similar, we only explain some of them below. Note also that we may assume that  $t = 0, 1, 2$  by Lemma 3.4.5, which simplifies the computations considerably.

As for  $H^{\bullet}(\mathcal{X}, \mathcal{C}_{44}(-t)) = H^{\bullet}(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-t))$  ( $0 \leq t \leq 2$ ), they vanish except in the case where  $t = 0$  since  $g$  is a  $\mathbb{P}^2$ -bundle and hence  $Rg_*^i \mathcal{O}_{\mathcal{X}}(-t) = 0$  for  $t = 1, 2$  and  $i \geq 0$ .  $H^{\bullet}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$  vanish except  $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$  by the Kodaira vanishing theorem.

As for  $H^{\bullet}(\mathcal{X}, \mathcal{C}_{34}(-t)) = H^{\bullet}(\mathcal{X}, g^*\mathcal{F}(-tH_{\mathcal{X}}))$  ( $0 \leq t \leq 2$ ), they vanish except in the case where  $t = 0$  by a similar reason. We have

$$H^{\bullet}(\mathcal{X}, g^*\mathcal{F}) \simeq H^{\bullet}(\mathrm{G}(2, V), \mathcal{F}),$$

which vanish except  $H^0(\mathrm{G}(2, V), \mathcal{F}) \simeq V^*$  by the Bott theorem 2.0.1.

As for  $H^{\bullet}(\mathcal{X}, \mathcal{C}_{24}(-t)) = H^{\bullet}(\mathcal{X}, \mathrm{Coker} \varphi(-t))$  ( $0 \leq t \leq 2$ ), consider the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathcal{X}}(-(t+1)H_{\mathcal{X}} + 2L_{\mathcal{X}}) \rightarrow \mathcal{S}^2(g^*\mathcal{F})(-t) \rightarrow \mathrm{Coker} \varphi(-t) \rightarrow 0.$$

Since  $g$  is a  $\mathbb{P}^2$ -bundle,  $H^{\bullet}(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-(t+1)H_{\mathcal{X}} + 2L_{\mathcal{X}}))$  vanishes except possibly in the case where  $t = 2$ , and  $H^{\bullet}(\mathcal{X}, \mathcal{S}^2(g^*\mathcal{F})(-t))$  vanishes except possibly in the case where  $t = 0$ . Since  $K_{\mathcal{X}} = -3H_{\mathcal{X}} - 2L_{\mathcal{X}}$ ,  $H^{\bullet}(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-3H_{\mathcal{X}} + 2L_{\mathcal{X}}))$  is Serre dual to  $H^{8-\bullet}(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(-4L_{\mathcal{X}}))$ , which is isomorphic to  $H^{8-\bullet}(\mathrm{G}(2, V), \mathcal{O}_{\mathrm{G}(2, V)}(-4))$ , and hence vanish. We have  $H^{\bullet}(\mathcal{X}, \mathcal{S}^2(g^*\mathcal{F})) \simeq H^{\bullet}(\mathrm{G}(2, V), \mathcal{S}^2\mathcal{F})$ , which vanish except  $H^0(\mathrm{G}(2, V), \mathcal{S}^2\mathcal{F}) \simeq \mathcal{S}^2V^*$  by the Bott theorem 2.0.1.  $\square$

The triangulated subcategory generated by the dual Lefschetz collection is contained in the derived category  $\mathcal{D}^b(\mathcal{X})$ ,

$$(3.10) \quad \langle \mathcal{D}_{\mathcal{X}}(-4), \dots, \mathcal{D}_{\mathcal{X}}(-1), \mathcal{D}_{\mathcal{X}} \rangle \subset \mathcal{D}^b(\mathcal{X}).$$

Since  $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$  is a resolution of rational singularities whose exceptional locus is an irreducible divisor  $E_f$ , we have a triangulated subcategory  $\hat{\mathcal{D}} \subset \mathcal{D}^b(\tilde{\mathcal{X}})$  called a categorical resolution of  $\mathcal{D}^b(\mathcal{X})$  for every dual Lefschetz decomposition of  $\mathcal{D}^b(E_f)$  [Ku3, Theorem 1]. There is a natural dual Lefschetz decomposition of  $\mathcal{D}(E_f)$  for the  $\mathbb{P}^3$ -fibration  $E_f \rightarrow \mathcal{X}_0 = v_2(\mathbb{P}^4)$  [Sa]. Then we can verify that the categorical resolution  $\tilde{\mathcal{D}}$  is strongly crepant since  $\mathcal{X}$  is Gorenstein and the dual Lefschetz decomposition is rectangular and also has length equal to the discrepancy of the resolution  $f$ .

It is expected that the subcategory (3.10) coincides with the categorical, strongly crepant resolution  $\tilde{\mathcal{D}}$ . Namely, the dual Lefschetz collection is expected to give the dual Lefschetz *decomposition* of the triangulated subcategory  $\tilde{\mathcal{D}}$ . It is also expected that  $\tilde{\mathcal{D}}$  is equivalent to the noncommutative resolution  $\mathcal{D}^b(\mathcal{X}, \mathcal{R})$  of  $\mathcal{D}^b(\mathcal{X})$  associated to the sheaf

$$\mathcal{R} := f_* \text{End}(\mathcal{O}_{\tilde{\mathcal{X}}} \oplus \mathcal{O}_{\tilde{\mathcal{X}}}(-L_{\tilde{\mathcal{X}}}).$$

Detailed study will be presented in a future publication [HoTa4].

In the next section, we will introduce the double cover  $\mathcal{Y}$  of the symmetroid  $\mathcal{H}$ . After making a nice resolution of  $\tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  in Subsection 6.6, we find a Lefschetz collection in the derived category  $\mathcal{D}^b(\tilde{\mathcal{Y}})$  in Theorem 8.1.1,

$$(3.11) \quad \langle \mathcal{D}_{\tilde{\mathcal{Y}}}, \mathcal{D}_{\tilde{\mathcal{Y}}}(1), \dots, \mathcal{D}_{\tilde{\mathcal{Y}}}(9) \rangle \subset \mathcal{D}^b(\tilde{\mathcal{Y}}).$$

Since the singularity of the double symmetroid  $\mathcal{Y}$  is complicated, the theorem [Ku3, Theorem 1] seems to be difficult to apply for the resolution  $\tilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  to obtain the categorical resolution  $\tilde{\mathcal{D}}$  of  $\mathcal{D}^b(\mathcal{Y})$ . However, we expect that the Lefschetz collection (3.11) gives a Lefschetz decomposition of a categorical crepant resolution, if exists, of  $\mathcal{D}^b(\mathcal{Y})$ . We present further discussions about the relations between the (dual) Lefschetz collections (3.10) and (3.11) at the end of the next subsection.

### 3.5. The Calabi-Yau manifold $X$ of a Reye congruence.

For generality of arguments below, consider  $\mathcal{X}$  for any  $n > 0$ . We choose a general linear subsystem  $P$  in  $|\mathcal{O}_{\mathcal{X}}(1)|$  of dimension  $n$ . We regard  $P$  as a general  $n$ -plane in  $\mathbb{P}(\mathbb{S}^2 V^*)$  associated with a linear subspace  $L \simeq \mathbb{C}^{n+1} \subset \mathbb{S}^2 V^*$ , i.e.,  $P = \mathbb{P}(L)$ . Explicitly we assume the form  $P = |Q_{A_1}, Q_{A_2}, \dots, Q_{A_{n+1}}|$  with the quadratic forms  $q_{A_i}$  ( $1 \leq i \leq n+1$ ) on  $\mathbb{P}(V)$ .

Let  $P^\perp = \mathbb{P}(L^\perp) \subset \mathbb{P}(\mathbb{S}^2 V)$  be the orthogonal projective space with  $L^\perp \subset \mathbb{S}^2 V$ , and define  $X := \mathcal{X} \cap P^\perp$ .  $X$  is naturally identified with the complete intersection in  $\mathcal{X} = \mathbb{S}^2 \mathbb{P}(V)$ ,

$$X = \{A_1 \cdot w_{xy} = \dots = A_{n+1} \cdot w_{xy} = 0\} \cap \mathcal{X}.$$

Since  $P$  is general, the orthogonal space  $P^\perp$  is disjoint from  $\text{Sing } \mathcal{X}$  from dimensional reason, and hence  $X$  is smooth by the Bertini theorem. Conversely several properties of  $X$  follows from assuming only that  $X$  is smooth; first of all, smoothness implies that  $P^\perp$  is disjoint from  $\text{Sing } \mathcal{X} = \mathcal{X}_0$ , hence we may consider  $X$  is embedded in  $\tilde{\mathcal{X}}$ . Moreover, we see the following properties:

**Proposition 3.5.1.** *The morphism  $g: \tilde{\mathcal{X}} \rightarrow \text{G}(2, V)$  gives the isomorphism  $X \cong g(X)$ . The linear system  $P$  of quadrics is free from base points.*

*Proof.* Both assertions follows from  $X \cap \mathcal{X}_0 = \emptyset$  in  $\mathcal{X}$ .

We start with the first assertion. Since  $X$  is fiberwise a linear section with respect to  $g$ , we have only to show that if a fiber  $\ell$  of  $g$  intersects  $X$ , then  $X \cap \ell$  consists of only one point. Assume the contrary. Then  $X \cap \ell$  is a linear subspace of  $\ell$  of positive dimension. Since  $E_f \cap \ell$  is a conic in  $\ell$ , we have  $E_f \cap X \neq \emptyset$ , a contradiction to that  $X \cap \mathcal{X}_0 = \emptyset$  in  $\mathcal{X}$ .

As for the second assertion, we note that the base locus of  $P$  is given by  $\{\mathbf{x} \mid A \cdot w_{\mathbf{x}\mathbf{x}} = {}^t\mathbf{x}A\mathbf{x} = 0 \text{ for any } [A] \in P\}$ , where  $A \cdot w_{\mathbf{x}\mathbf{x}} = {}^t\mathbf{x}A\mathbf{x}$  follows from (3.2). This is empty if and only if  $X \cap \mathcal{X}_0 = \emptyset$ .  $\square$

The *Reye congruence* is a line congruence in  $G(2, V)$ , which is nothing but the isomorphic image of  $X$  under  $g : \tilde{\mathcal{X}} \rightarrow G(2, V)$  (cf. [Ol]). In this paper we often identify  $X$  with  $g(X)$ .

Below, we present a characterization of  $X \subset G(2, V)$  by the projective geometry of quadrics in  $P$ . For a point  $[l] \in G(2, V)$ , we denote by  $P_l$  the subspace of  $P$  which parameterizes quadrics in  $\mathbb{P}(V)$  containing the line  $l$ , namely, if we write  $l = \langle \mathbf{x}, \mathbf{y} \rangle$  with  $\mathbf{x}, \mathbf{y} \in \mathbb{P}(V)$ , then  $P_l = \{[A] \in P \mid {}^t\mathbf{x}A\mathbf{y} = {}^t\mathbf{x}A\mathbf{x} = {}^t\mathbf{y}A\mathbf{y} = 0\}$ .

**Proposition 3.5.2.** *A point  $[l] \in G(2, V)$  is contained in  $X \subset G(2, V)$  if and only if  $P_l$  is an  $(n-2)$ -plane.*

*Proof.* First we show that, for any  $[l] \in G(2, V)$ ,  $\dim P_l \leq n-2$ . Assume by contradiction that  $\dim P_l \geq n-1$ . Then all the quadrics  $[Q_A] \in P$  contains  $l$  or the restrictions of quadrics  $[Q_A] \in P$  to  $l$  reduce to a unique quadric on  $l$ . This is a contradiction to the second statement of Proposition 3.5.1.

Assume that  $[l] \in X$ . We have only to show  $\dim P_l \geq n-2$ . By Proposition 3.5.1, there exists a unique  $w_{\mathbf{x}\mathbf{y}} \in X \subset \mathcal{X}$  such that  $l = \langle \mathbf{x}, \mathbf{y} \rangle$ . Since  $X = \mathcal{X} \cap P^\perp$ , we have  ${}^t\mathbf{x}A\mathbf{y} = A \cdot w_{\mathbf{x}\mathbf{y}} = 0$  for any  $[A] \in P$  by (3.2). Therefore  $P_l = \{[A] \in P \mid {}^t\mathbf{x}A\mathbf{x} = {}^t\mathbf{y}A\mathbf{y} = 0\}$ , and hence  $\dim P_l \geq n-2$ .

Conversely, assume that  $\dim P_l = n-2$ . We may choose a basis  $A_1, \dots, A_{n+1}$  of  $P$  such that  $A_1, \dots, A_{n-1}$  form a basis of  $P_l$ . Then the restrictions of quadrics  $[Q_A] \in P$  to  $l$  form the pencil of quadrics on  $l$  spanned by  $Q_{A_n}|_l$  and  $Q_{A_{n+1}}|_l$ . The corresponding pencil of symmetric  $(1, 1)$ -divisors on  $l \times l$  has two base points  $(\mathbf{x}, \mathbf{y})$  and  $(\mathbf{y}, \mathbf{x})$  with  $\mathbf{x} \neq \mathbf{y}$  since the base points of this pencil are disjoint from the diagonal. Then note that  $l = \langle \mathbf{x}, \mathbf{y} \rangle$ . By the definition of  $(\mathbf{x}, \mathbf{y})$ , we have  ${}^t\mathbf{x}A_n\mathbf{y} = {}^t\mathbf{x}A_{n+1}\mathbf{y} = 0$ . By the choice of  $A_1, \dots, A_{n-1}$ , we have  ${}^t\mathbf{x}A_1\mathbf{y} = \dots = {}^t\mathbf{x}A_{n-1}\mathbf{y} = 0$ . Therefore  ${}^t\mathbf{x}A\mathbf{y} = 0$  for any  $[A] \in P$ . This implies that  $w_{\mathbf{x}\mathbf{y}} \in X$  by (3.2).  $\square$

When  $n$  is even,  $X$  is a Calabi-Yau  $(n-1)$ -fold and satisfies:

**Proposition 3.5.3.**  *$\pi_1(X) \simeq \mathbb{Z}_2$  and  $\text{Pic } X \simeq \mathbb{Z} \oplus \mathbb{Z}_2$ , where the free part of  $\text{Pic } X$  is generated by the class  $D$  of a hyperplane section of  $\mathcal{X}$  restricted to  $X$ .*

*Proof.* Consider the complete intersection  $\tilde{X}$  in  $\mathbb{P}(V) \times \mathbb{P}(V)$  defined by the pull-back of  $P$ . Then we have the projection morphism  $\pi_{\tilde{X}} : \tilde{X} \rightarrow X$ .

By the Lefschetz theorem,  $\pi_1(\tilde{X}) = \{1\}$ . Since the map  $\pi_{\tilde{X}}$  is an étale double cover, we have  $\pi_1(X) \simeq \mathbb{Z}_2$ .

Let  $E$  be any effective divisor on  $X$ . Since  $\pi_X^*E$  is  $\mathbb{Z}_2$ -invariant, it is of type  $(m, m)$  with some non-negative integer  $m$ . We may choose a homogeneous equation  $F_E$  of  $\pi_X^*E$  as symmetric or skew-symmetric. If  $F_E$  is symmetric, then  $E \sim mD$ . Assume that  $F_E$  is skew-symmetric. Let  $\tilde{D}$  be a skew-symmetric  $(1, 1)$ -divisor. Then  $\pi_X^*E - m\tilde{D} = \text{div}(\alpha)$ , where  $\alpha$  is a  $\mathbb{Z}_2$ -invariant rational function of  $\tilde{X}$ . Then  $\alpha$  is the pull-back of a rational function  $\beta$  of  $X$ , and we see that  $E - m\pi_{\tilde{X}*}\tilde{D} = \text{div}(\beta)$  by looking at the zero and pole of  $\beta$ . Therefore  $\text{Pic } X$  is generated by the classes of  $D$  and  $\pi_{\tilde{X}*}\tilde{D}$ . Since  $2D \sim 2\pi_{\tilde{X}*}\tilde{D}$ , we conclude that  $\text{Pic } X \simeq \mathbb{Z} \oplus \mathbb{Z}_2$ .  $\square$

When  $n = 4$ ,  $X$  is a Calabi-Yau threefold with the following invariants [HoTa1, Proposition 2.1]:

$$\deg(X) = 35, \quad c_2.D = 50, \quad h^{2,1}(X) = 26, \quad h^{1,1}(X) = 1,$$

where  $c_2$  is the second Chern class of  $X$ .

#### 4. THE DOUBLE SYMMETROID $\mathcal{Y}$ AND THE CALABI-YAU VARIETY $Y$

Consider a Calabi-Yau  $(n-1)$ -fold  $X$  of Reye congruence for arbitrary even  $n$ . Under the projective duality (3.4), we find that  $X$  is naturally paired with another Calabi-Yau variety  $H$  in the symmetroid  $\mathcal{H}$  (see [HoTa1] for  $n=4$ ). The geometry of the symmetroid  $\mathcal{H}$  was studied in detail by Tjurin [Tj]. In particular, he defined a double cover  $\mathcal{Y} \rightarrow \mathcal{H}$ , which we call *double symmetroid*. In this section, we elaborate Tjurin's construction. By considering linear sections of this double cover, we define a Calabi-Yau variety  $Y$  for arbitrary even  $n$ . For  $n=4$ ,  $Y$  turns out to be a smooth Calabi-Yau threefold. We will remark that the Calabi-Yau threefolds  $X$  and  $Y$  arise naturally from the (dual) Lefschetz collections (3.10) and (3.11) assuming the projective homological duality [Ku2, Ku4] in our case.

##### 4.1. The resolution $\mathcal{U}$ of $\mathcal{H}$ .

Let us define

$$\mathcal{U} := \{(\mathbf{x}, A) \mid A\mathbf{x} = \mathbf{0}\} \subset \mathbb{P}(V) \times \mathbb{P}(\mathbb{S}^2 V^*).$$

The variety  $\mathcal{U}$  parameterizes pairs  $(\mathbf{x}, A)$  such that  $\mathbf{x}$  is contained in the singular locus of the quadric  $Q_A$ . The image of  $\mathcal{U}$  by the natural projection  $\text{pr}_2: \mathbb{P}(V) \times \mathbb{P}(\mathbb{S}^2 V^*) \rightarrow \mathbb{P}(\mathbb{S}^2 V^*)$  is nothing but the locus of singular symmetric matrices, which is the symmetric determinantal hypersurface (symmetroid)  $\mathcal{H}$ . The morphism  $\mathcal{U} \rightarrow \mathcal{H}$  is one to one over the locus of matrices of rank  $n$  since the singular locus of a quadric of rank  $n$  consists of one point. Moreover, by a similar reason, the fiber is isomorphic to  $\mathbb{P}^{n-i}$  over a point in the locus of matrices of rank  $i$ . The natural projection  $\text{pr}_1: \mathbb{P}(V) \times \mathbb{P}(\mathbb{S}^2 V^*) \rightarrow \mathbb{P}(V)$  represents  $\mathcal{U}$  as a projective bundle over  $\mathbb{P}(V)$ , where the fiber  $\mathcal{U}_x$  over a point  $x$  of  $\mathbb{P}(V)$  is the space of singular quadrics in  $\mathbb{P}(V)$  containing  $x$  in their singular loci. In particular  $\mathcal{U}$  is smooth. We can regard  $\mathcal{U}_x$  as the space of quadrics in  $\mathbb{P}^{n-1}$ , where  $\mathbb{P}^{n-1}$  is the image of the projection  $\mathbb{P}(V) \dashrightarrow \mathbb{P}^{n-1}$  from  $x$ .

**Proposition 4.1.1.**  $\mathcal{U} \simeq \mathbb{P}(\mathbb{S}^2(\Omega(1)))$ .

*Proof.* Let  $[V_1] \in \mathbb{P}(V)$  be a point. Since the Euler sequence of  $\mathbb{P}(V)$

$$0 \rightarrow \mathcal{O}(-1) \rightarrow V \otimes \mathcal{O} \rightarrow T(-1) \rightarrow 0,$$

restricts over the point  $[V_1]$  to  $0 \rightarrow V_1 \rightarrow V \rightarrow V/V_1 \rightarrow 0$ , we see that the fiber of  $\Omega(1)$  at  $[V_1]$  is  $(V/V_1)^*$ , which parameterizes hyperplanes of  $\mathbb{P}(V)$  through  $[V_1]$ . Therefore the fiber of  $\mathbb{P}(\Omega(1)) \rightarrow \mathbb{P}(V)$  over  $[V_1]$  parameterizes hyperplanes of  $\mathbb{P}(V)$  through  $[V_1]$ . The fiber of  $\mathbb{P}(\mathbb{S}^2(\Omega(1))) \rightarrow \mathbb{P}(V)$  over  $[V_1]$  parameterizes quadrics in  $\mathbb{P}(V)$  which are given as the sum of the products of hyperplanes through  $[V_1]$ , namely, quadrics in  $\mathbb{P}(V)$  singular at  $[V_1]$ .  $\square$

We have obtained the following diagram:

$$(4.1) \quad \begin{array}{ccc} & \mathcal{U} = \mathbb{P}(\mathbb{S}^2(\Omega(1))) & \\ \swarrow \pi_{\mathcal{U}} & & \searrow \text{pr}_2 \\ \mathbb{P}(V) & & \mathcal{H}. \end{array}$$



#### 4.2. The double covering $\mathcal{Y}$ of $\mathcal{H}$ .

Here we construct the double cover  $\mathcal{Y}$  of  $\mathcal{H}$ , the *double symmetroid*, by considering  $\frac{n}{2}$ -planes contained in each singular quadric.

Let us first consider a variety  $\mathcal{Z}$  which parameterizes the pair of quadrics  $Q$  and  $\frac{n}{2}$ -planes  $\mathbb{P}(\Pi)$  such that  $\mathbb{P}(\Pi) \subset Q$ . To parametrize  $\frac{n}{2}$ -planes in  $\mathbb{P}(V)$ , consider the Grassmannian  $G(\frac{n+2}{2}, V)$ . Let

$$(4.2) \quad 0 \rightarrow \mathcal{W}^* \rightarrow V^* \otimes \mathcal{O}_{G(\frac{n+2}{2}, V)} \rightarrow \mathcal{U} \rightarrow 0$$

be the dual of the universal exact sequence on  $G(\frac{n+2}{2}, V)$ , where  $\mathcal{W}$  is the universal quotient bundle of rank  $\frac{n}{2}$  and  $\mathcal{U}$  is the dual of the universal subbundle of rank  $\frac{n+2}{2}$ . For an  $\frac{n}{2}$ -plane  $\mathbb{P}(\Pi) \subset \mathbb{P}(V)$ , there exists a natural surjection  $S^2 V^* \rightarrow S^2 H^0(\mathbb{P}(\Pi), \mathcal{O}_{\mathbb{P}(\Pi)}(1))$  such that the projectivization of the kernel consisting of the quadrics containing  $\mathbb{P}(\Pi)$ . By relativizing this surjection over  $G(\frac{n+2}{2}, V)$ , we obtain the following surjection:  $S^2 V^* \otimes \mathcal{O}_{G(\frac{n+2}{2}, V)} \rightarrow S^2 \mathcal{U}$ . Let  $\mathcal{E}^*$  be the kernel of this surjection, and consider the following exact sequence:

$$(4.3) \quad 0 \rightarrow \mathcal{E}^* \rightarrow S^2 V^* \otimes \mathcal{O}_{G(\frac{n+2}{2}, V)} \rightarrow S^2 \mathcal{U} \rightarrow 0.$$

Set  $\mathcal{Z} = \mathbb{P}(\mathcal{E}^*)$  and denote by  $\rho_{\mathcal{Z}}$  the projection  $\mathcal{Z} \rightarrow G(\frac{n+2}{2}, V)$ . By (4.3),  $\mathcal{Z}$  is contained in  $G(\frac{n+2}{2}, V) \times \mathbb{P}(S^2 V^*)$ . Since the fiber of  $\mathcal{E}^*$  over  $[\Pi]$  parameterizes quadrics in  $\mathbb{P}(V)$  containing  $\mathbb{P}(\Pi)$ , we have

$$\mathcal{Z} = \{([\Pi], [Q]) \mid \mathbb{P}(\Pi) \subset Q\} \subset G(\frac{n+2}{2}, V) \times \mathbb{P}(S^2 V^*).$$

Note that  $Q$  in  $([\Pi], [Q]) \in \mathcal{Z}$  is a singular quadric since a smooth quadric does not contain  $\frac{n}{2}$ -planes. Hence the symmetroid  $\mathcal{H}$  is the image of the natural projection  $\mathcal{Z} \rightarrow \mathbb{P}(S^2 V^*)$ . Now we introduce

$$\mathcal{Z} \xrightarrow{\pi_{\mathcal{Z}}} \mathcal{Y} \xrightarrow{\rho_{\mathcal{Y}}} \mathcal{H},$$

the Stein factorization of  $\mathcal{Z} \rightarrow \mathcal{H}$ . By (4.3), the tautological divisor  $H_{\mathbb{P}(\mathcal{E}^*)}$  of  $\mathbb{P}(\mathcal{E}^*) \rightarrow G(\frac{n+2}{2}, V)$  is nothing but the pull-back of a hyperplane section of  $\mathcal{H}$ . We set

$$M_{\mathcal{Z}} := H_{\mathbb{P}(\mathcal{E}^*)} = \pi_{\mathcal{Z}}^* \circ \rho_{\mathcal{Y}}^* \mathcal{O}_{\mathcal{H}}(1).$$

We denote by  $\mathcal{Z}_{[Q]}$  the fiber of  $\mathcal{Z} \rightarrow \mathcal{H}$  over a point  $[Q] \in \mathcal{H}$ .

**Lemma 4.2.1.** *For a quadric  $Q$  of rank  $n$ , the fiber  $\mathcal{Z}_{[Q]}$  is the disjoint union of two copies of orthogonal Grassmann  $\text{OG}(\frac{n}{2}, n)$ .*

*Proof.* The quadric  $Q$  of rank  $n$  induces a non-degenerate symmetric bilinear form  $q$  on the quotient  $V/V_1$ , where  $V_1$  is the 1-dimensional vector space such that  $[V_1]$  is the vertex of  $Q$ . Then  $\frac{n}{2}$ -planes on  $Q$  naturally correspond to the maximal isotropic subspaces in  $V/V_1$  with respect to  $q$ , which are parameterized by the disjoint union of two copies of orthogonal Grassmann  $\text{OG}(\frac{n}{2}, n)$ .  $\square$

**Proposition 4.2.2.** *The morphism  $\mathcal{Y} \rightarrow \mathcal{H}$  is of degree two and is branched along the locus of quadrics of rank less than or equal to  $n - 1$ .*

*Proof.* By Lemma 4.2.1, the degree of  $\mathcal{Y} \rightarrow \mathcal{H}$  is two since  $\mathcal{Z}_{[Q]}$  has two connected components for a quadric  $Q$  of rank  $n$ . If a quadric  $Q$  has rank less than or equal to  $n - 1$ , the family of  $\frac{n}{2}$ -planes in  $Q$  is connected. Hence we have the assertion.  $\square$

By this proposition, we see that  $\mathcal{Y}$  parameterizes connected families of  $\frac{n}{2}$ -planes in singular quadrics in  $\mathbb{P}(V)$  (cf. Fig.1).

**Definition 4.2.3.** Related to the morphism  $\rho_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{H}$ , we define  $G_{\mathcal{Y}}$  to be the inverse image by  $\rho_{\mathcal{Y}}$  of the locus of quadrics of rank less than or equal to  $n - 2$ .  $\square$

Since  $G_{\mathcal{Y}}$  is contained in the ramification locus of  $\rho_{\mathcal{Y}}$ , it is clearly isomorphic to the locus of quadrics of rank less than or equal to  $n - 2$ .

$\mathcal{Y}$  has the following nice properties in view of the minimal model program.

**Proposition 4.2.4.** *The Picard number of  $\mathcal{Z}$  is two and  $\pi_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathcal{Y}$  is a Mori fiber space. In particular,  $\mathcal{Y}$  is a  $\mathbb{Q}$ -factorial Gorenstein canonical Fano variety with Picard number one. Moreover,  $\mathcal{Y}$  is smooth outside  $G_{\mathcal{Y}}$  and the Fano index of  $\mathcal{Y}$  is  $\frac{n(n+1)}{2}$ .*

*Proof.* Since  $\mathcal{Z}$  is a projective bundle over  $G(\frac{n+2}{2}, V)$ , its Picard number is two. Note that the relative Picard number of  $\pi_{\mathcal{Z}} : \mathcal{Z} \rightarrow \mathcal{Y}$  is one, then we see that the Picard number of  $\mathcal{Y}$  is one. Since a general fiber of  $\pi_{\mathcal{Z}}$  is a Fano variety by Lemma 4.2.1,  $\pi_{\mathcal{Z}}$  is a Mori fiber space. By [KMM, Lemma 5-1-5],  $\mathcal{Y}$  is  $\mathbb{Q}$ -factorial, and, by [F-o, Corollary 4.6],  $\mathcal{Y}$  has only kawamata log terminal singularities.  $\mathcal{Y}$  is a Gorenstein Fano variety since it is a double cover of a Gorenstein Fano variety  $\mathcal{H}$  and the ramification locus has codimension greater than one by Proposition 4.2.2. Thus  $\mathcal{Y}$  has only canonical singularities.

As we mentioned in Subsection 3.2, the singular locus of  $\mathcal{H}$  is the locus of quadrics of rank less than or equal to  $n - 1$ . Since  $\rho_{\mathcal{Y}} : \mathcal{Y} \rightarrow \mathcal{H}$  is étale outside this locus,  $\mathcal{Y}$  is smooth outside the inverse image of this locus. Moreover, by the proof of [HoTa1, Lemma 3.6],  $\mathcal{H}$  has only ordinary double points along the locus of quadrics of rank  $n - 1$ . Since  $\rho_{\mathcal{Y}}$  is ramified along this locus,  $\mathcal{Y}$  is smooth along the inverse image of this locus. Therefore  $\mathcal{Y}$  is smooth outside  $G_{\mathcal{Y}}$ .

Since  $K_{\mathcal{H}} = \mathcal{O}_{\mathcal{H}}(\frac{n(n+1)}{2})$ , the Fano index of  $\mathcal{Y}$  is  $\frac{n(n+1)}{2}$ .  $\square$

*Remark.* We will show that  $\mathcal{Y}$  has only terminal singularities when  $n = 4$  (see Proposition 6.9.2).  $\square$

When  $n = 4$ , we have more detailed descriptions of the fibers of  $\rho_{\mathcal{Y}}$ .

**Proposition 4.2.5.** *If  $\text{rank } Q = 4$ , then  $\mathcal{Z}_{[Q]}$  is a disjoint union of two smooth rational curves. Each connected component is identified with a conic on  $G(3, V)$ . If  $\text{rank } Q = 3$ , then  $\mathcal{Z}_{[Q]}$  is a smooth rational curve, which is also identified with a conic on  $G(3, V)$ . If  $\text{rank } Q = 2$ , then  $\mathcal{Z}_{[Q]}$  is the union of two  $\mathbb{P}^3$ 's intersecting at one point. If  $\text{rank } Q = 1$ , then  $\mathcal{Z}_{[Q]}$  is a (non-reduced)  $\mathbb{P}^3$ .*

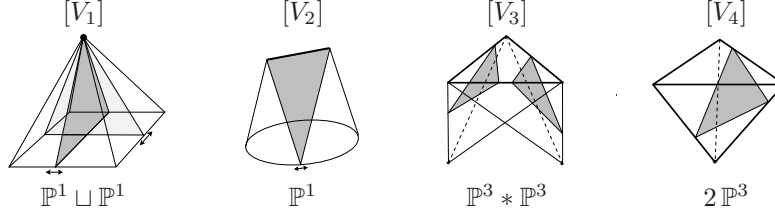
*Proof.* If  $\text{rank } Q = 4$ , the fiber  $\mathcal{Z}_{[Q]}$  consists of two disconnected components, and is isomorphic to the orthogonal Grassmann  $\text{OG}(2, 4)$  by Lemma 4.2.1. To be more explicit, let  $[V_1] \in \mathbb{P}(V)$  be the vertex of  $Q$ . Then the quadric  $Q$  is the cone over  $\mathbb{P}^1 \times \mathbb{P}^1$  with the vertex  $[V_1]$ . There are two distinct  $\mathbb{P}^1$ -families of lines on  $\mathbb{P}^1 \times \mathbb{P}^1$ .

Each of the families can be understood as a corresponding conic on  $G(2, V/V_1)$ , which gives one of the connected components of  $OG(2, 4)$ . Under the natural map  $G(2, V/V_1) \rightarrow G(3, V)$ , we have a  $\mathbb{P}^1$  family of 2-planes on  $Q$  parameterized by a conic on  $G(3, V)$ .

If  $\text{rank } Q = 3$ , the vertex of the quadric  $Q$  is a line  $[V_2] \subset \mathbb{P}(V)$ . The quadric  $Q$  is the cone over a conic with the vertex  $[V_2]$ . The conic is contained in  $\mathbb{P}(V/V_2) = G(1, V/V_2)$ , and can be identified with a conic in  $G(3, V)$  under the natural map  $G(1, V/V_2) \rightarrow G(3, V)$ .

If  $\text{rank } Q = 2$ , then the quadric  $Q$  has a vertex  $[V_3] \subset \mathbb{P}(V)$  and is the union of two 3-planes intersecting along the 2-plane  $[V_3]$ . Hence  $\mathcal{Z}_{[Q]} \subset G(3, V)$  is given by the union of the corresponding  $\mathbb{P}^3$ 's, i.e.,  $G(3, 4)$ 's in  $G(3, V)$ , which intersect at one point  $[V_3]$ .

If  $\text{rank } Q = 1$ , then  $Q$  is a double 3-plane. Thus  $\mathcal{Z}_{[Q]}$  is a (non-reduced)  $\mathbb{P}^3 \cong G(3, 4)$ .  $\square$



**Fig.1. Quadrics  $Q$  in  $\mathbb{P}(V)$  and families of planes therein.**

The singular loci of  $Q$  are written by  $[V_k]$  with  $k = 5 - \text{rk } Q$ . Also the parameter spaces of the planes in each  $Q$  are shown. See also Fig.3 in Subsection 6.9.

We write by  $G_{\mathcal{Y}}^1$  (resp.  $G_{\mathcal{Y}}^2$ ) the inverse image under  $\rho_{\mathcal{Y}}$  of the locus of quadrics of rank one (resp. two). We see that  $G_{\mathcal{Y}} \simeq S^2\mathbb{P}(V^*)$ ,  $G_{\mathcal{Y}}^1 \simeq v_2(\mathbb{P}(V^*))$  and  $G_{\mathcal{Y}}^2 = G_{\mathcal{Y}} \setminus G_{\mathcal{Y}}^1$ . Using these, we summarize our construction above for  $n = 4$  in the following diagram:

$$\begin{array}{ccccc}
 & & \mathcal{Z} & \xrightarrow{\mathbb{P}^8\text{-bundle}} & G(3, V) \\
 & & \downarrow \pi_{\mathcal{Z}} & \searrow \rho_{\mathcal{Z}} & \\
 (4.4) & G_{\mathcal{Y}}^1 & \subset & G_{\mathcal{Y}} & \subset & \mathcal{Y} \\
 & \wr & & \wr & & \downarrow \rho_{\mathcal{Y}} \\
 & v_2(\mathbb{P}(V)) & \subset & S^2\mathbb{P}(V) & \subset & \mathcal{H},
 \end{array}$$

where  $\pi_{\mathcal{Z}}$  is a  $\mathbb{P}^1$ -fibration over  $\mathcal{Y} \setminus G_{\mathcal{Y}}$  by Proposition 4.2.5. In Section 6, studying the birational geometry of  $\mathcal{Y}$ , we will find a nice desingularization  $\widetilde{\mathcal{Y}}$  of  $\mathcal{Y}$ . We will study the geometry of  $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  along the loci  $G_{\mathcal{Y}}$  and  $G_{\mathcal{Y}}^1$  in full details in Subsection 6.9 and Section 7.

### 4.3. The Calabi-Yau variety $Y$ .

Assume that a Reye congruence Calabi-Yau  $(n-1)$ -fold  $X = \mathcal{X} \cap P^\perp$  is given by a general  $n$ -plane  $P$  in  $\mathbb{P}(S^2V^*)$ , i.e.,  $P = \mathbb{P}(L)$  with  $L \simeq \mathbb{C}^{n+1} \subset S^2V^*$ . Define  $H := \mathcal{H} \cap P$ . Then  $H$  is a determinantal hypersurface of degree  $n+1$  in  $P \simeq \mathbb{P}(L)$  and hence the canonical bundle of  $H$  is trivial.

Let  $Y$  be the pull-back of  $H$  on  $\mathcal{Y}$ . According to [Ku2], we call  $Y$  is *orthogonal* to  $X$  and vice versa.

**Proposition 4.3.1.**  *$Y$  is a Calabi-Yau variety.*

*Proof.* The canonical divisor  $K_Y$  is trivial since  $K_H \sim 0$  and the branch locus of  $Y \rightarrow H$  has the codimension greater than or equal to two.

By Proposition 4.2.4,  $\mathcal{Y}$  has only Gorenstein canonical singularities and  $h^i(\mathcal{O}_{\mathcal{Y}}) = 0$  for  $i > 0$ . Then, by a version of the Bertini theorem,  $Y$  also has Gorenstein canonical singularities. It is standard to derive  $h^i(\mathcal{O}_Y) = 0$  for  $0 < i < \dim Y$  from  $h^i(\mathcal{O}_{\mathcal{Y}}) = 0$  for  $i > 0$  by the Kodaira-Kawamata-Viewheg vanishing theorem.  $\square$

In the rest of this paper, we restrict our attention to  $n = 4$ . In this case,  $Y$  is smooth by Proposition 4.2.4 since  $Y \cap G_{\mathcal{Y}} = \emptyset$  by dimensional reason. We will show at the end of Subsection 6.9 that this Calabi-Yau manifold  $Y$  coincides with the double covering  $Y$  defined in [HoTa1], where  $Y$  is called as the (*shifted*) *Mukai dual* to  $X$ .

In the previous paper [HoTa1, Prop.3.11 and Prop.3.12], we have determined invariants of the Calabi-Yau threefold  $Y$ . Here we reproduce these invariants using the construction summarized in (4.4).

Let us first recall (4.3) for the definition of  $\mathcal{E}^*$  and set  $\mathcal{E} := (\mathcal{E}^*)^*$ . Then we have

**Lemma 4.3.2.**  $c_1(\mathcal{E}) = c_1(\mathcal{O}_{G(3,V)}(4))$ .

*Proof.* Note that  $c_1(\mathcal{O}_{G(3,V)}(1))$  is given by the Schubert cycle  $\sigma_1$ , which is  $c_1(\mathcal{U})$  in our notation. Since  $\text{rk } \mathcal{U} = 3$ , we have  $c_1(\mathcal{E}) = c_1(S^2\mathcal{U}) = 4c_1(\mathcal{U})$ . Thus we have the assertion.  $\square$

Now let us note the relative Euler sequence associated with the projective bundle  $\rho_{\mathcal{X}}: \mathcal{Z} = \mathbb{P}(\mathcal{E}^*) \rightarrow G(3,V)$ :

$$(4.5) \quad 0 \rightarrow \mathcal{O}_{\mathcal{Z}} \rightarrow \rho_{\mathcal{Z}}^* \mathcal{E}^*(M_{\mathcal{Z}}) \rightarrow T_{\mathcal{Z}} \rightarrow \rho_{\mathcal{Z}}^* T_{G(3,V)} \rightarrow 0.$$

From this we obtain the following:

**Lemma 4.3.3.** *Let  $N_{\mathcal{Z}} := \rho_{\mathcal{Z}}^* \mathcal{O}_{G(3,V)}(1)$ .  $K_{\mathcal{Z}} = -9M_{\mathcal{Z}} - N_{\mathcal{Z}}$  holds for the canonical divisor  $K_{\mathcal{Z}}$  on  $\mathcal{Z}$ , and we have the following cohomologies for the sheaves on  $\mathcal{Z}$  and for  $0 \leq k \leq 10$ :*

- (1)  $H^\bullet(\mathcal{O}_{\mathcal{Z}}(-(k+1)M_{\mathcal{Z}} + N_{\mathcal{Z}})) = 0$ .
- (2)  $H^\bullet(\mathcal{O}_{\mathcal{Z}}(-kM_{\mathcal{Z}})) \simeq H^\bullet(\rho_{\mathcal{Z}}^* \mathcal{E}^*(-(k-1)M_{\mathcal{Z}})) \simeq \begin{cases} \mathbb{C} & (\bullet = 0, k = 0) \\ \mathbb{C} & (\bullet = 13, k = 10) \\ 0 & (\text{otherwise}) \end{cases}.$
- (3)  $H^\bullet(\rho_{\mathcal{Z}}^* T_{G(3,V)}(-kM_{\mathcal{Z}})) \simeq H^\bullet(T_{\mathcal{Z}}(-kM_{\mathcal{Z}})) \simeq \begin{cases} \mathbb{C}^{24} & (\bullet = 0, k = 0) \\ \mathbb{C} & (\bullet = 12, k = 10) \\ 0 & (\text{otherwise}) \end{cases}.$

*Proof.* The claimed formula of  $K_{\mathcal{X}}$  follows by taking the determinant of the Euler sequence (4.5). We should note that  $\text{rank } \mathcal{E} = 9$  and  $N_{\mathcal{X}} = \rho_{\mathcal{X}}^* \mathcal{O}_{G(3,V)}(1)$ .

For the calculations of the cohomologies in (1), (2), (3), we use the Serre duality, the Kodaira vanishing theorem and also the Bott theorem 2.0.1 as well as the defining exact sequence (4.3) of  $\mathcal{E}^*$ .

(1) By the Serre duality, we have  $H^\bullet(\mathcal{O}_{\mathcal{X}}(-(k+1)M_{\mathcal{X}} + N_{\mathcal{X}})) \simeq H^{14-\bullet}(\mathcal{O}_{\mathcal{X}}((k-8)M_{\mathcal{X}} - 2N_{\mathcal{X}}))$ . From this, the claimed vanishings follow for the range  $0 \leq k \leq 7$  for all  $\bullet$  since  $\rho_{\mathcal{X}}: \mathcal{X} \rightarrow G(3, V)$  is a  $\mathbb{P}^8$ -bundle. When  $k = 8$ , the vanishing follows from  $H^{14-\bullet}(\mathcal{O}_{\mathcal{X}}(-2N_{\mathcal{X}})) \simeq H^{14-\bullet}(G(3, V), \mathcal{O}_{G(3,V)}(-2)) = 0$ . When  $k = 9$ , we need to evaluate  $H^{14-\bullet}(\mathcal{O}_{\mathcal{X}}(M_{\mathcal{X}} - 2N_{\mathcal{X}})) \simeq H^{14-\bullet}(G(3, V), \mathcal{E}(-2))$ . By tensoring the dual of (4.3) by  $\mathcal{O}_{G(3,V)}(-2)$  and using the Bott theorem, it is easy to obtain the claimed vanishing. When  $k = 10$ , we have  $H^{14-\bullet}(\mathcal{O}_{\mathcal{X}}(2M_{\mathcal{X}} - 2N_{\mathcal{X}})) \simeq H^{14-\bullet}(G(3, V), S^2\mathcal{E}(-2))$ . For the calculation of the cohomologies of  $S^2\mathcal{E}(-2) = S^2\mathcal{E} \otimes \mathcal{O}_{G(3,V)}(-2)$ , we introduce the sheaf  $\mathcal{K}$  which fits into the exact sequences in the following two ways:

$$0 \rightarrow \mathcal{K} \rightarrow S^2(S^2V) \otimes \mathcal{O}_{G(3,V)} \rightarrow S^2\mathcal{E} \rightarrow 0, \quad 0 \rightarrow S^2(S^2\mathcal{U}^*) \rightarrow \mathcal{K} \rightarrow S^2\mathcal{U}^* \otimes \mathcal{E} \rightarrow 0.$$

By using the formula  $S^2(S^2\mathcal{U}^*) = \Sigma^{(2,2,0)}\mathcal{U}^* \oplus \Sigma^{(4,0,0)}\mathcal{U}^*$  (see [FH, p.189] for example) and the Bott theorem, the calculations are straightforward.

(2) Since the calculations of  $H^\bullet(\mathcal{O}_{\mathcal{X}}(-kM_{\mathcal{X}}))$  is easy, we omit them. For the cohomologies  $H^\bullet(\rho_{\mathcal{X}}^*\mathcal{E}^*(-(k-1)M_{\mathcal{X}}))$ , when  $k = 0$ , we have to evaluate  $H^\bullet(\rho_{\mathcal{X}}^*\mathcal{E}^*(M_{\mathcal{X}})) = H^\bullet(G(3, V), \mathcal{E}^* \otimes \mathcal{E})$ . This can be done by considering two short exact sequences; one from tensoring the defining exact sequence (4.3) by  $\mathcal{E}$  and the other from tensoring the dual of (4.3) by  $S^2\mathcal{U}$ . The cases of other values of  $k$  are rather easy, so we omit their details.

(3) For the calculations of  $H^\bullet(\rho_{\mathcal{X}}^*T_{G(3,V)}(-kM_{\mathcal{X}}))$ , we use  $T_{G(3,V)} = \mathcal{U} \otimes \mathcal{W}$ . For example, for  $k = 0$ , we evaluate  $H^\bullet(\rho_{\mathcal{X}}^*T_{G(3,V)}) = H^\bullet(G(3, V), \mathcal{U} \otimes \mathcal{W})$ , which is non-vanishing only for  $\bullet = 0$  with the result  $\Sigma^{(1,0,0,0,-1)}V^* \simeq \mathbb{C}^{\oplus 24}$ . For  $k \geq 1$ , use the Serre duality and the defining exact sequence (4.3).

Finally the calculations of  $H^\bullet(T_{\mathcal{X}}(-kM_{\mathcal{X}}))$  are done with the relative Euler sequence (4.5) and also using the results obtained so far. Since they are straightforward, we omit them here.  $\square$

Let  $M$  be the pull-back of  $\mathcal{O}_H(1)$  to  $Y$ . The following proposition refines the results in [HoTa1, Propositions 3.11 and 3.12]:

**Proposition 4.3.4.** *The 3-fold  $Y$  is a simply connected Calabi-Yau 3-fold such that  $\text{Pic } Y = \mathbb{Z}[M]$ ,  $M^3 = 10$ ,  $c_2(Y) \cdot M = 40$  and  $e(Y) = -50$ . In particular,  $h^{1,1}(Y) = 1$  and  $h^{1,2}(Y) = 26$ .*

*Proof.* We have already shown that  $Y$  is a smooth Calabi-Yau threefold. Since  $Y \rightarrow H$  is a double cover, we have  $M^3 = 2c_1(\mathcal{O}_H(1))^3 = 10$ .

To calculate other invariants, we use the restriction of  $\pi_{\mathcal{X}}: \mathcal{X} \rightarrow \mathcal{Y}$  over  $Y$ , which is a  $\mathbb{P}^1$ -fibration. Set  $Z := \pi_{\mathcal{X}}^{-1}(Y)$  and  $\pi_Z$  be the restriction of  $\pi_{\mathcal{X}}$  to  $Z$ . We also set  $N_Z$  and  $M_Z$  be the restrictions of  $N_{\mathcal{X}}$  and  $M_{\mathcal{X}}$  to  $Z$ , respectively.

Let us first note that we have  $K_Z = M_Z - N_Z$  for the canonical divisor by Lemma 4.3.3, since  $Z$  is a complete intersection of ten members of  $|M_{\mathcal{X}}|$ . Also we note the following Koszul resolution of  $\mathcal{O}_Z$  as a  $\mathcal{O}_{\mathcal{X}}$ -module:

$$(4.6) \quad 0 \rightarrow \wedge^{10}\{\mathcal{O}_{\mathcal{X}}(-M_{\mathcal{X}})^{\oplus 10}\} \rightarrow \cdots \rightarrow \mathcal{O}_{\mathcal{X}}(-M_{\mathcal{X}})^{\oplus 10} \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_Z \rightarrow 0.$$

We observe the following isomorphisms:

$$(4.7) \quad H^\bullet(Z, T_Z) \simeq H^\bullet(Z, \pi_Z^* T_Y) \simeq H^\bullet(Y, T_Y),$$

where we also note that  $H^\bullet(Y, T_Y)$  vanishes for  $\bullet = 0, 3$  since  $Y$  is a Calabi-Yau threefold. The second isomorphism follows from the fact that  $Z \rightarrow Y$  is a  $\mathbb{P}^1$ -fibration. To see the first isomorphism, let us note the exact sequence  $0 \rightarrow T_{Z/Y} \rightarrow T_Z \rightarrow \pi_Z^* T_Y \rightarrow 0$ , from which we have  $T_{Z/Y} = \mathcal{O}_Z(-K_Z)$  since  $K_Y = 0$  and  $T_{Z/Y}$  is an invertible sheaf. Then we have  $H^\bullet(Z, T_{Z/Y}) = H^\bullet(Z, \mathcal{O}_Z(-M_Z + N_Z))$ . Tensoring the resolution (4.6) by  $\mathcal{O}(-M_{\mathcal{X}} + N_{\mathcal{X}})$  and using (1) of Lemma 4.3.3, we see the vanishing  $H^\bullet(Z, T_{Z/Y}) = 0$ . This entails the first isomorphism.

Next let us consider the exact sequence

$$(4.8) \quad 0 \rightarrow T_Z \rightarrow T_{\mathcal{X}}|_Z \rightarrow \mathcal{O}_Z(M_Z)^{\oplus 10} \rightarrow 0.$$

Since  $Z \rightarrow Y$  is a  $\mathbb{P}^1$ -fibration, we have  $H^\bullet(Z, \mathcal{O}_Z(M_Z)) \simeq H^\bullet(Y, \mathcal{O}_Y(M))$ , where the r.h.s. vanish by the Kodaira vanishing theorem except  $H^0(Y, \mathcal{O}_Y(M)) \simeq \mathbb{C}^5$ . Therefore, by (4.7) and (4.8), we have

$$(4.9) \quad \begin{aligned} & H^\bullet(T_Z) \simeq H^\bullet(T_{\mathcal{X}}|_Z) \text{ for } \bullet \geq 2 \\ 0 \rightarrow H^0(T_{\mathcal{X}}|_Z) \rightarrow \mathbb{C}^{\oplus 50} \rightarrow H^1(T_Z) \rightarrow H^1(T_{\mathcal{X}}|_Z) \rightarrow 0. \end{aligned}$$

We finally calculate the cohomology of the sheaf  $T_{\mathcal{X}}|_Z$  as

$$(4.10) \quad H^0(T_{\mathcal{X}}|_Z) = \mathbb{C}^{\oplus 24}, \quad H^2(T_{\mathcal{X}}|_Z) = \mathbb{C}, \quad H^i(T_{\mathcal{X}}|_Z) = 0 \ (i \neq 0, 2).$$

These results follow by tensoring the resolution (4.6) by  $T_{\mathcal{X}}$  and using (3) of Lemma 4.3.3. Now combining (4.10) with (4.9) and (4.7), we obtain  $h^1(T_Z) = h^1(T_Y) = 26$  and  $h^2(T_Z) = h^2(T_Y) = 1$ .

Since  $Y$  is a Calabi-Yau threefold, we have  $e(Y) = 2(h^2(T_Y) - h^1(T_Y)) = -50$ . Also by the Riemann-Roch formula, and the vanishings derived above, we have

$$\chi(Y, \mathcal{O}_Y(M)) = \frac{M^3}{6} + \frac{c_2(Y) \cdot M}{12} = \dim H^0(Y, \mathcal{O}_Y(M)) = 5.$$

From this, we obtain  $c_2(Y) \cdot M = 40$ .

It remains to show the simple-connectedness of  $Y$ . This will imply  $\text{Pic } Y \simeq \mathbb{Z}$  since we have already shown  $\rho(Y) = h^2(T_Y) = 1$ . Let  $W$  be a general 4-dimensional complete intersection of  $\mathcal{Y}$  by members of  $|M_{\mathcal{Y}}|$  containing  $Y$ . Then  $W$  is a Fano 4-fold, and moreover, since  $\mathcal{Y}$  is smooth outside  $G_{\mathcal{Y}}$  by Proposition 4.2.4 and the codimension of  $G_{\mathcal{Y}}$  in  $\mathcal{Y}$  is 5,  $W$  is smooth. Therefore it is a well-known fact that  $W$  is simply connected and then so is  $Y$  by the Lefschetz theorem since  $Y$  is an ample divisor on  $Z$ .  $\square$

The Calabi-Yau threefold  $X = \mathcal{X} \cap \mathbb{P}(L^\perp)$  is defined by fixing a general 4-plane  $\mathbb{P}(L)$  in  $|\mathcal{O}_{\mathcal{X}}(1)| = \mathbb{P}(\mathcal{S}^2 V^*)$ . Corresponding to  $X$ , the orthogonal Calabi-Yau threefold  $Y$  is defined by the pull-back of the determinantal quintic  $H = \mathcal{H} \cap \mathbb{P}(L)$  to the double symmetroid  $\mathcal{Y}$ . We will write this pull-back by  $Y = \mathcal{Y} \cap \mathbb{P}(L)$  identifying the general 4-plane  $\mathbb{P}(L)$  in  $|\mathcal{O}_{\mathcal{H}}(1)|$  with the corresponding 4-plane in  $|\rho_{\mathcal{Y}}^* \mathcal{O}_{\mathcal{H}}(1)|$ .

This construction of the orthogonal linear sections is extended to general  $(r-1)$ -planes  $\mathbb{P}(L_r)$  with  $L_r \simeq \mathbb{C}^r$  ( $r = 1, \dots, 14$ ), for which we define the corresponding linear sections by  $\mathcal{X}_{L_r} = \mathcal{X} \cap \mathbb{P}(L_r^\perp)$  and  $\mathcal{Y}_{L_r} = \mathcal{Y} \cap \mathbb{P}(L_r)$ .

Our (dual) Lefschetz collections (3.10) and (3.11) are in accord with the so-called homological projective duality due to Kuznetsov [Ku2] between  $\mathcal{X}$  and  $\mathcal{Y}$ . To see this, let us assume (3.10) and (3.11) give the (dual) Lefschetz decompositions of the noncommutative resolutions  $\mathcal{D}^b(\mathcal{X}, \mathcal{R})$  and  $\mathcal{D}^b(\mathcal{Y}, \mathcal{R}')$ , respectively. Assume also that  $\mathcal{Y}$  is homological projective dual to  $\mathcal{X}$ , then by following [Ku2] the noncommutative resolutions of the linear sections  $\mathcal{D}^b(\mathcal{X}_{L_r}, \mathcal{R})$ ,  $\mathcal{D}^b(\mathcal{Y}_{L_r}, \mathcal{R}')$  are described by

$$\mathcal{D}^b(\mathcal{X}_{L_r}, \mathcal{R}) = \begin{cases} \langle \mathcal{D}_{\mathcal{X}}(-4), \dots, \mathcal{D}_{\mathcal{X}}(-1) \rangle & (r = 1) \\ \langle \mathcal{D}_{\mathcal{X}}(-(5-r)), \dots, \mathcal{D}_{\mathcal{X}}(-1), \mathcal{D}^b(\mathcal{Y}_{L_r}, \mathcal{R}') \rangle & (2 \leq r \leq 5) \end{cases}$$

and

$$\mathcal{D}^b(\mathcal{Y}_{L_r}, \mathcal{R}') = \begin{cases} \langle \mathcal{D}^b(\mathcal{X}, \mathcal{R}), \mathcal{D}_{\widetilde{\mathcal{Y}}}(1), \dots, \mathcal{D}_{\widetilde{\mathcal{Y}}}(r-5) \rangle & (5 \leq r \leq 8) \\ \langle \mathcal{D}_{\widetilde{\mathcal{Y}}}(1), \dots, \mathcal{D}_{\widetilde{\mathcal{Y}}}(r-5) \rangle & (9 \leq r \leq 14) \end{cases}$$

where  $\mathcal{D}^b(\mathcal{X}_{L_r}, \mathcal{R}) = \mathcal{D}^b(\mathcal{X}_{L_r})$  for  $r > \dim(\text{Sing}(\mathcal{X}))$  and also  $\mathcal{D}^b(\mathcal{Y}_{L_r}, \mathcal{R}') = \mathcal{D}^b(\mathcal{Y}_{L_r})$  for  $r < \dim(\text{Sing}(\mathcal{Y})) = 7$ . In particular, when  $r = 5$ , this indicates the derived equivalence  $D^b(X) \simeq D^b(Y)$  for the Reye congruence  $X$  and the double covering  $Y$ .

We will find that there is a nice desingularization  $\widetilde{\mathcal{Y}}$  (Subsection 6.6). Constructing the noncommutative resolution  $\mathcal{D}^b(\mathcal{Y}, \mathcal{R}')$  seems to be a difficult task since the divisorial contraction  $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  studied in Subsection 6.9 does not satisfy the conditions of [Ku3, Theorem 4.4]. However, in the derived category  $\mathcal{D}^b(\widetilde{\mathcal{Y}})$ , we will find the Lefschetz collection (3.11) in Theorem 8.1.1, which supports the homological projective duality above. Also, in a separate publication [HoTa3], we will show the derived equivalence  $D^b(X) \simeq D^b(Y)$  by finding a certain ideal sheaf  $\mathcal{I}$  in  $\mathcal{D}(\widetilde{\mathcal{X}} \times \widetilde{\mathcal{Y}})$  which defines a Fourier-Mukai functor. It turns out that the ideal sheaf  $\mathcal{I}$  has a locally free resolution in terms of the sheaves in Theorems 3.4.4 and 8.1.1.



## 5. THE HILBERT SCHEME OF CONICS $\mathcal{H}_0$ ON $G(3, V)$

Recall that we are restricting our attention to the case of  $n = 4$  ( $\dim V = 5$ ).

By Proposition 4.2.5, the parameter space of a connected family of planes in a quadric of rank 3 or 4 is a smooth conic in  $G(3, V)$  (see also the proof of Proposition 5.3.1 below). This fact naturally leads us to consider the Hilbert scheme of conics on  $G(3, V)$ ,

$$\mathcal{H}_0 := \text{Hilb}^{\text{conic}} G(3, V),$$

which has dimension 13 (see Proposition 5.3.1). In [IM, §3.1], the scheme  $\mathcal{H}_0$  is studied in detail. In this section, we review and refine the results of [ibid.] for our purposes in the subsequent sections.

### 5.1. Planes and conics on $G(3, V)$ .

There are several types of conics on  $G(3, V)$ . To describe them, let us first classify the types of planes contained in  $G(3, V)$ . It is well known that there are two types of such planes:

i) planes which are written by

$$P_{V_2} := \{[\Pi] \in G(3, V) \mid V_2 \subset \Pi\} \cong \mathbb{P}^2$$

for some  $V_2$ , or

ii) planes which are written by

$$P_{V_1 V_4} := \{[\Pi] \in G(3, V) \mid V_1 \subset \Pi \subset V_4\} \cong \mathbb{P}^2$$

for some  $V_1$  and  $V_4$  with  $V_1 \subset V_4$ .

**Definition 5.1.1.** We call a plane  $P_{V_2}$  a  $\rho$ -plane and a plane  $P_{V_1 V_4}$  a  $\sigma$ -plane.  $\square$

For a conic  $q$  in  $\mathbb{P}(\wedge^3 V)$ , there exists a unique plane  $\mathbb{P}_q^2 \subset \mathbb{P}(\wedge^3 V)$  such that  $q \in \mathbb{P}_q^2$ . If  $q$  is contained in  $G(3, V)$ , namely,  $[q]$  is an element of the Hilbert scheme  $\mathcal{H}_0$ , then there are two possibilities:  $\mathbb{P}_q^2 \subset G(3, V)$  or  $\mathbb{P}_q^2 \not\subset G(3, V)$ .

When  $\mathbb{P}_q^2 \subset G(3, V)$  we call  $q$  a  $\rho$ -conic if  $\mathbb{P}_q^2 = P_{V_2}$  for some  $V_2$ , and  $\sigma$ -conic if  $\mathbb{P}_q^2 = P_{V_1 V_4}$  for some  $V_1$  and  $V_4$  following [IM, §3.1]. When  $\mathbb{P}_q^2 \not\subset G(3, V)$ , we have  $q = \mathbb{P}_q^2 \cap G(3, V)$  and we call such a conic a  $\tau$ -conic.

**Example 5.1.2.** Taking a basis  $\mathbf{e}_1, \dots, \mathbf{e}_5$  of  $V$ , choose the subspaces, for example, to be  $V_1 = \langle \mathbf{e}_4 \rangle$ ,  $V_4 = \langle \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \rangle$  and  $V_2 = \langle \mathbf{e}_4, \mathbf{e}_5 \rangle$ . Then typical examples of  $\tau$ -,  $\sigma$ -,  $\rho$ -conics, respectively, may be given explicitly in terms of the homogeneous coordinates of  $G(3, V)$ ;

$$q_\tau = \left\{ \begin{bmatrix} s & t & 0 & 0 & 0 \\ 0 & 0 & s & t & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\}, \quad q_\sigma = \left\{ \begin{bmatrix} s & t & 0 & 0 & 0 \\ 0 & s & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \right\}, \quad q_\rho = \left\{ \begin{bmatrix} s^2 & st & t^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right\},$$

where  $[s, t] \in \mathbb{P}^1$  parameterizes each conic  $q$ . The  $\tau$ -conic above is on a unique plane  $\mathbb{P}_{q_\tau}^2 = \{[p_{135}, p_{145}, p_{235}, p_{245}] \mid p_{145} = p_{235}\} \subset \mathbb{P}(\wedge^3 V)$  and characterized by the conic  $\mathbb{P}_{q_\tau}^2 \cap G(3, V) = \{p_{135}p_{245} - p_{145}^2 = 0\}$ , where  $p_{ijk}$  are the Plücker coordinates. The  $\sigma$ -conic above is on the plane  $P_{V_1 V_4} \subset G(3, V)$ . It is easy to see  $P_{V_1 V_4} = \{[p_{124}, p_{134}, p_{234}]\} \subset \mathbb{P}(\wedge^3 V)$ , and the equation  $p_{124}p_{234} - p_{134}^2 = 0$  for  $q_\sigma$ . Similarly, we see that the  $\rho$ -conic above is on a plane  $P_{V_2} = \{[p_{145}, p_{245}, p_{345}]\}$  with its equation  $p_{145}p_{345} - p_{245}^2 = 0$ .

### 5.2. The $\mathrm{SL}(V)$ -orbit decomposition of $\mathcal{B}_0$ .

It is easy to see that  $\mathrm{SL}(V)$ -orbits of  $\mathcal{B}_0$  are labelled by their types of conics  $\rho$ ,  $\sigma$  and  $\tau$ , and their ranks 1, 2 and 3 ([IM, §3.1]). Below we list these orbits. We denote by  $\Gamma_*$  the locus of  $*$ -conics in  $\mathcal{B}_0$ .

- i)  $\rho$ -planes are parameterized by  $G(2, V)$  since a  $\rho$ -plane is given by  $P_{V_2}$  with a line  $\mathbb{P}(V_2)$  in  $\mathbb{P}(V)$ . Moreover, the total space of the family of  $\rho$ -planes is naturally identified with  $\mathbb{P}(\mathcal{G})$  where  $\mathcal{G}$  is the universal quotient bundle of  $G(2, V)$ . Therefore  $\Gamma_\rho$ , consisting the conics in the  $\rho$ -planes, is isomorphic to  $\mathbb{P}(\mathcal{S}^2\mathcal{G}^*)$ , hence  $\Gamma_\rho$  is of codimension 2 in  $\mathcal{B}_0$ .

$\Gamma_\rho$  splits into three orbits consisting of  $\rho$ -conics of rank 1, 2 and 3, and they have codimension 5, 3 and 2, respectively.

- ii) Since pairs  $(V_1, V_4)$  such that  $V_1 \subset V_4$  are parameterized by the flag variety  $F(1, 4, V) \simeq \mathbb{P}(\Omega_{\mathbb{P}(V)}^1) \simeq \mathbb{P}(\Omega_{\mathbb{P}(V^*)}^1)$ , and conics in  $P_{V_1V_4}$  are parameterized by  $\mathbb{P}^5$ , the locus  $\Gamma_\sigma$  of  $\sigma$ -conics are  $\mathbb{P}^5$ -bundle over  $F(1, 4, V)$ , which is a divisor in  $\mathcal{B}_0$ .

$\Gamma_\sigma$  splits into three orbits consisting of conics of rank 1, 2 and 3, and they have codimension 4, 2 and 1, respectively.

- iii)  $\Gamma_\tau$  splits into three orbits consisting of conics of rank 1, 2 and 3, respectively.  $\tau$ -conics of rank 3 form the open orbit of  $\mathcal{B}_0$ .

Let  $q$  be a  $\tau$ -conic of rank 2. Then  $q$  is a pair of intersecting lines, say,  $l_1$  and  $l_2$ . We may write  $l_i = \{[\Pi] \mid V_2^{(i)} \subset \Pi \subset V_4^{(i)}\}$ , where  $\mathbb{P}(V_2^{(i)}) \subset \mathbb{P}(V)$  are lines and  $\mathbb{P}(V_4^{(i)}) \subset \mathbb{P}(V)$  are 3-planes for  $i = 1, 2$ . Since  $l_1 \cap l_2 \neq \emptyset$ , it holds  $\dim(V_2^{(1)} \cap V_2^{(2)}) \geq 1$ . Since  $q$  is not a  $\rho$ -conic,  $V_2^{(1)} \neq V_2^{(2)}$ . Therefore we see that  $\dim(V_2^{(1)} \cap V_2^{(2)}) = 1$  and  $l_1 \cap l_2 = [V_2^{(1)} + V_2^{(2)}]$ . If  $V_4^{(1)} = V_4^{(2)}$ , then  $q$  is contained in  $P_{V_2^{(1)} \cap V_2^{(2)}, V_4^{(1)}}$  and then  $q$  is a  $\sigma$ -conic, a contradiction. Therefore  $V_4^{(1)} \neq V_4^{(2)}$ . In summary, we have the following conditions:

$$V_2^{(1)} \neq V_2^{(2)}, V_4^{(1)} \neq V_4^{(2)}, V_4^{(1)} \cap V_4^{(2)} = V_2^{(1)} + V_2^{(2)}.$$

By this description and simple dimension counts, we see that the orbit of  $\tau$ -conics of rank 2 has codimension 1 in  $\mathcal{B}_0$ . Compare this description of conics of rank 2 with the constructions in Subsection 7.2.

As for the orbit of  $\tau$ -conics of rank 1, we will see that it has codimension 3 in  $\mathcal{B}_0$  in Corollary 7.1.3.

**Example 5.2.1.** We present examples of  $\tau$ -,  $\sigma$ -, and  $\rho$ -conics of rank 2, respectively.

- (1) From the above description, an example of  $\tau$ -conic of rank 2 may be given by  $q_\tau = l_1 \cup l_2$  with

$$l_1 = \{[\mathbf{e}_1, \mathbf{e}_2, s\mathbf{e}_3 + t\mathbf{e}_4 \mid [s, t] \in \mathbb{P}^1\}, \quad l_2 = \{[\mathbf{e}_2, \mathbf{e}_3, u\mathbf{e}_1 + v\mathbf{e}_5 \mid [u, v] \in \mathbb{P}^1\}.$$

It is easy to see that this conic is on a unique plane  $\mathbb{P}_{q_\tau}^2 = \{[p_{123}, p_{124}, p_{235}]\} \subset \mathbb{P}(\wedge^3 V)$ . Then the equation of  $q_\tau$  can be read as  $\mathbb{P}_{q_\tau} \cap G(3, V) = \{p_{124}p_{235} = 0\}$ .

- (2) Take  $V_1, V_2, V_4$  as in Example 5.1.2. Then  $P_{V_1V_4} = \{[p_{124}, p_{134}, p_{234}]\} \subset G(3, V)$  and  $P_{V_2} = \{[p_{145}, p_{245}, p_{345}]\} \subset G(3, V)$ . As an example of  $\sigma$ -conic of rank 2 on the  $\sigma$ -plane  $P_{V_1V_4}$ , we may have  $q_\sigma = l_3 \cup l_4$  with

$$l_3 = \{[s\mathbf{e}_1 + t\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \mid [s, t] \in \mathbb{P}^1\}, \quad l_4 = \{[\mathbf{e}_1, u\mathbf{e}_2 + v\mathbf{e}_3, \mathbf{e}_4 \mid [u, v] \in \mathbb{P}^1\}.$$

The equation of this conic is  $p_{124}p_{234} = 0$ . Similarly, as a  $\rho$ -conic of rank 2 on the  $\rho$ -plane  $P_{V_2}$ , we may have  $q_\sigma = l_5 \cup l_6$  with

$$l_5 = \{[se_1 + te_2, e_4, e_5] \mid [s, t] \in \mathbb{P}^1\}, \quad l_6 = \{[ue_2 + ve_3, e_4, e_5] \mid [u, v] \in \mathbb{P}^1\}.$$

The equation of this conic is  $p_{145}p_{345} = 0$ .

### 5.3. The birational map $\mathcal{Y}_0 \dashrightarrow \mathcal{Y}$ .

We denote by  $\mathcal{U}^*$  the universal subbundle on  $G(3, V)$ , and regard  $\mathbb{P}(\mathcal{U}^*)$  as the universal family of the projective planes in  $\mathbb{P}(V)$ . There is a natural projection  $\varphi_{\mathcal{U}} : \mathbb{P}(\mathcal{U}^*) \rightarrow \mathbb{P}(V)$ .

Suppose a *smooth* conic  $q \subset G(3, V)$  is given. Note that  $\mathcal{U}$  on  $G(3, V)$  restricts as  $\mathcal{U}|_q \simeq \mathcal{O}(1)_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}$ , or  $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$  since  $\mathcal{U}$  is generated by its global sections and  $\deg \mathcal{U}|_q = c_1(\wedge^3 \mathcal{U}|_q) = 2$ . Let  $S$  be the image of  $\mathbb{P}(\mathcal{U}^*|_q)$  under  $\varphi_{\mathcal{U}}$ . Then there are two possibilities; 1) the degree of  $\mathbb{P}(\mathcal{U}^*|_q) \rightarrow S$  is two and  $S$  is a 3-plane, i.e., a quadric of rank 1, or 2) the degree of  $\mathbb{P}(\mathcal{U}^*|_q) \rightarrow S$  is one and  $S$  is a quadric of rank 4 or 3 according to the type of  $\mathcal{U}|_q$ , i.e.,  $\mathcal{O}(1)_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}$ , or  $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}$ , respectively (see Example 5.1.2). If  $S$  is a 3-plane  $\mathbb{P}(V_4)$ , then  $q \subset \{[\Pi] \in G(3, V) \mid \Pi \subset V_4\}$  and  $\mathbb{P}_q^2$  must be a  $\sigma$ -plane according to the classification of the planes in  $G(3, V)$ . Hence  $q$  is a  $\sigma$ -conic.

**Proposition 5.3.1.**  *$\varphi_{\mathcal{U}}$  induces a birational map  $\mathcal{Y}_0 \dashrightarrow \mathcal{Y}$ . Under this birational map, smooth  $\tau$ -,  $\rho$ -conics are mapped to one of the connected components of  $\mathcal{Z}_{[Q]}$  for a quadric  $Q$  of rank 4 or 3, respectively (cf. Proposition 4.2.5).*

*Proof.* Recall that  $\text{codim } \Gamma_\tau = 0$ ,  $\text{codim } \Gamma_\rho = 2$  and  $\text{codim } \Gamma_\sigma = 1$  for smooth conics. Then, for the assertions, we have only to show that for a smooth  $\tau$ -conic (resp.  $\rho$ -conic)  $q \subset G(3, V)$ , all the planes  $\mathbb{P}(\Pi)$  belonging to  $q$  sweep a quadric of rank 4 (resp. rank 3) in  $\mathbb{P}^4$ . These are exactly the properties we have seen above.  $\square$

Indeed, we will show that the birational map  $\mathcal{Y}_0 \dashrightarrow \mathcal{Y}$  above is a morphism (see Subsection 6.8). Hence  $\mathcal{Y}_0$  is a natural desingularization of  $\mathcal{Y}$ .  $\mathcal{Y}_0$  is, however, slightly too large for our purpose and hence we will construct a smaller desingularization  $\widehat{\mathcal{Y}}$  of  $\mathcal{Y}$  by studying birational geometries of  $\mathcal{Y}_0$  in detail (see Subsection 6.6).

### 5.4. Explicit descriptions of $\mathcal{Y}_0$ .

In this subsection, we show that  $\mathcal{Y}_0$  is birational to a Grassmann bundle  $\mathcal{Y}_3$  over  $\mathbb{P}(V)$  and describe  $\mathcal{Y}_0$  from the geometry of  $\mathcal{Y}_3$ .

We first note that we can identify  $G(2, V/V_1)$  with the subset  $\{[\Pi] \in G(3, V) \mid V_1 \subset \Pi\}$ , and accordingly, we can identify a conic  $q \subset G(2, V/V_1)$  with a conic  $q \subset G(3, V)$  satisfying  $V_1 \subset \Pi$  for all  $[\Pi] \in q$ . Following [IM, §3.1], we set

$$\begin{aligned} \mathcal{Y}_1 &= \{ (q, [V_1]) \mid q \subset G(2, V/V_1), [V_1] \in \mathbb{P}(V) \} \\ &\equiv \{ (q, [V_1]) \mid q \subset G(3, V) \text{ s.t. } V_1 \subset \Pi \text{ for all } [\Pi] \in q; [V_1] \in \mathbb{P}(V) \}. \end{aligned}$$

**Proposition 5.4.1.** *For a conic  $q \subset G(3, V)$  which is not a  $\tau$ -double line, all the planes  $[\Pi]$  on  $q$  intersect at a non-empty set. In other words, there exists at least one point  $[V_1] \in \mathbb{P}(V)$  for which  $q \subset G(3, V)$  descends to  $q \subset G(2, V/V_1)$ .*

*Proof.* If  $q$  is a  $\sigma$ -conic or a  $\rho$ -conic, then the assertion is clear by definition. Assume that  $q$  is a smooth  $\tau$ -conic. Then, by Proposition 5.3.1, the image  $\varphi_{\mathcal{U}}(\mathbb{P}(\mathcal{U}^*|_q))$  is a quadric  $Q$  of rank 4, and the vertex  $v$  of  $Q$  has the desired property. Assume that  $q$  is a pair of distinct intersecting lines  $l_1$  and  $l_2$  in  $G(3, V)$ . As in the description of  $\Gamma_\tau$  in Subsection 5.2, we may write  $l_i = \{[\Pi] \mid V_2^{(i)} \subset \Pi \subset V_4^{(i)}\}$  ( $i = 1, 2$ ) with  $\dim(V_2^{(1)} \cap V_2^{(2)}) = 1$ , hence all  $[\Pi] \in q$  contain  $\mathbb{P}(V_2^{(1)} \cap V_2^{(2)}) \neq \emptyset$ .  $\square$

By this proposition, the natural projection morphism  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_0$  is dominant, hence is surjective since it is proper. In particular, as a consequence, even for  $q$  being a  $\tau$ -double line, there exists a  $[V_1] \in \mathbb{P}(V)$  such that  $q \subset G(2, V/V_1)$ . We will use this fact in Proposition 7.1.2.

Recall that  $G(2, V/V_1)$  is embedded in  $\mathbb{P}(\wedge^2 V/V_1)$  by the Plücker embedding. Following [IM, §3.1], we set

$$\mathcal{Y}_3 = \{(P_2, [V_1]) \mid P_2 \cong \mathbb{P}^2 \text{ is a plane in } \mathbb{P}(\wedge^2(V/V_1))\}.$$

There exists a natural morphism  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_3$  sending  $(q, [V_1])$  to  $(\mathbb{P}_q^2, [V_1])$ . Since for a plane  $P_2 \subset \mathbb{P}(\wedge^2(V/V_1))$ , the intersection  $G(2, V/V_1) \cap P_2$  is a conic in  $G(2, V/V_1)$  in general (it is  $P_2$  itself in special cases), the morphism  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_3$  is birational.

In summary, we have constructed the following diagram:

$$(5.1) \quad \begin{array}{ccccc} & & \mathcal{Y}_1 & \supset ([V_1], q) & (q \subset G(2, V/V_1)) \\ & \swarrow & & \searrow & \\ (\mathbb{P}_q^2, [V_1]) \in & \mathcal{Y}_3 & & \mathcal{Y}_0 & \supset q \\ & \downarrow \pi_{\mathcal{Y}_3} & & & \\ & [V_1] \in \mathbb{P}(V). & & & \end{array}$$

More details of this diagram are in order:

- i) For every conic  $[q] \in \mathcal{Y}_0$  except for  $\rho$ -conics, there exists a unique point  $[V_1] \in \mathbb{P}(V)$  such that  $q \subset G(3, V)$  descends to  $q \subset G(2, V/V_1)$ . Therefore the morphism  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_0$  is isomorphic outside  $\Gamma_\rho$ . Actually  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_0$  is the blow-up along  $\Gamma_\rho$  by [IM, Remark in the end of §3.1] (this corresponds to the blow-up  $\tilde{\rho}_{\mathcal{Y}_2}$  described in Subsection 6.6).
- ii) Consider  $G(2, V/V_1)$  in  $\mathbb{P}(\wedge^2(V/V_1))$  and define the following locus in  $\mathcal{Y}_3$ :

$$\mathcal{P} := \{(P_2, [V_1]) \mid P_2 \cong \mathbb{P}^2 \text{ is a plane in } G(2, V/V_1)\},$$

which is the disjoint union of two orthogonal Grassmann bundles. If  $P_2$  is not a plane in  $G(2, V/V_1)$ , then the intersection  $P_2 \cap G(2, V/V_1)$  determines a conic in  $G(2, V/V_1)$ . Therefore the morphism  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_3$  is isomorphic outside  $\mathcal{P}$ . Actually  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_3$  is the blow-up along  $\mathcal{P}$  by [IM, Remark in the end of §3.1].  $\mathcal{P}$  is the disjoint union of the image of the exceptional divisor over  $\Gamma_\rho$  of  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_0$  and the image of the strict transform of  $\Gamma_\sigma$  (see Fig.2), which we denote by

$$\mathcal{P} = \mathcal{P}_\rho \sqcup \mathcal{P}_\sigma.$$

This decomposition will be studied further in Subsection 6.3.

- iii) The fiber of  $\mathcal{Y}_3 \rightarrow \mathbb{P}(V)$  over a point  $[V_1] \in \mathbb{P}(V)$  is the set of all planes in  $\mathbb{P}(\wedge^2(V/V_1))$ , which is nothing but  $G(3, \wedge^2(V/V_1)) \simeq G(3, 6)$ . Since the Euler sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(V)}(-1) \rightarrow V \otimes \mathcal{O}_{\mathbb{P}(V)} \rightarrow T(-1) \rightarrow 0$$

restricts at the point  $[V_1]$  to  $0 \rightarrow V_1 \rightarrow V \rightarrow V/V_1 \rightarrow 0$ , we can identify the fibration  $\mathcal{Y}_3 \rightarrow \mathbb{P}(V)$  with the  $G(3, 6)$ -bundle over  $\mathbb{P}(V)$ ,  $\mathcal{Y}_3 = G(3, T(-1)^{\wedge 2})$ .

Putting the diagrams (4.1), (4.4), (5.1) and Proposition 5.3.1 together, we obtain the following diagram with some additional data:

$$(5.2) \quad \begin{array}{ccccc} & \mathcal{Z}_1 & & \mathcal{Z}_0 & \\ & \downarrow & \searrow & \downarrow \pi_{\mathcal{Z}_0} & \searrow \rho_{\mathcal{Z}_0} \\ & \mathcal{Y}_1 & & \mathcal{Y}_0 & \\ \swarrow & & \searrow & & \downarrow \pi_{\mathcal{Z}} \\ \mathcal{Y}_3 & & \mathcal{U} & \dashrightarrow & \mathcal{Y} \\ \downarrow \pi_{\mathcal{Y}_3} & \swarrow & \searrow & & \downarrow \\ \mathbb{P}(V) & & \mathcal{H} & & \end{array} \quad \begin{array}{c} \mathcal{Z} \xrightarrow{\rho_{\mathcal{Z}}} G(3, V), \\ \downarrow \pi_{\mathcal{Z}} \end{array}$$

where  $\pi_{\mathcal{Z}_0}: \mathcal{Z}_0 \rightarrow \mathcal{Y}_0$  is the universal family of conics,  $\mathcal{Z}_1 \rightarrow \mathcal{Y}_1$  is the base change of  $\pi_{\mathcal{Z}_0}$ , and  $\rho_{\mathcal{Z}_0}: \mathcal{Z}_0 \rightarrow G(3, V)$  is the natural projection.

Note that  $\mathcal{Z}_0 \rightarrow \mathcal{Y}_0$  is birationally equivalent to  $\mathcal{Z} \rightarrow \mathcal{Y}$  (cf. Propositions 4.2.5 and 5.3.1).

6. BIRATIONAL GEOMETRY OF  $\mathcal{Y}$  — MOLE'S NEST —

In this section, we extend the birational geometry of  $\mathcal{Y}$  summarized in (5.2) to the big diagram (6.1) below. By playing the two ray game starting from the Grassmann bundle  $\mathcal{Y}_3 = G(3, T(-1)^{\wedge 2}) \rightarrow \mathbb{P}(V)$ , we construct a Sarkisov link and, as its final step, obtain a divisorial contraction  $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ , which gives a nice resolution of  $\mathcal{Y}$ . The main steps of the two ray game are described in details in Subsections 6.5, 6.6 and also Subsection 6.9. In Fig.2, we schematically draw the entire picture of the birational geometries related to the resolution  $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ .

We also prepare several auxiliary results, which will be required in the subsequent sections. In particular, three locally free sheaves  $\widetilde{S}_L, \widetilde{Q}, \widetilde{T}$  on  $\mathcal{Y}$  are introduced in Definitions 6.7.1 and 6.7.3. Using these sheaves we will obtain a Lefschetz collection in  $\mathcal{D}^b(\widetilde{\mathcal{Y}})$  (Section 8).

$$(6.1) \quad \begin{array}{ccccccc} & & & \mathcal{Y}_2 & & \mathcal{Y}_0 & \\ & & & \downarrow \rho_{\mathcal{Y}_2} & \searrow p_{\mathcal{Y}_2} & \downarrow & \\ & & & \mathcal{Y}_3 \dashrightarrow \mathcal{U} & \searrow \tilde{\rho}_{\mathcal{Y}_2} & \downarrow & \mathcal{Y} \\ & \swarrow \rho_G & \searrow \pi_G & \downarrow \pi_{\mathcal{Y}_3} & \swarrow \pi_{\mathcal{U}} & \searrow \rho_{\widetilde{\mathcal{Y}}} & \downarrow \rho_{\mathcal{Y}} \\ G(3, V) & & \mathbb{P}(V) & & & & \mathcal{H} \end{array}$$

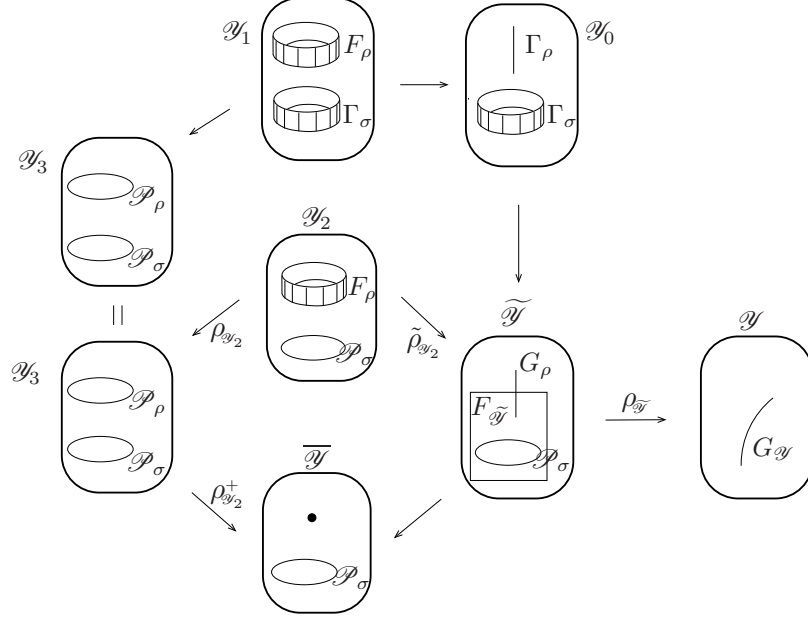
In the above diagram, all the varieties have natural  $\mathrm{SL}(V)$ -actions and all the morphisms are  $\mathrm{SL}(V)$ -equivariant. For convenience, we list the notation used in the diagram (see also Fig.2):

- $\mathcal{Y}_3 = G(3, T(-1)^{\wedge 2})$ :  $G(3, 6)$ -bundle over  $\mathbb{P}(V)$ .
- $\mathcal{Y}_2 \rightarrow \mathcal{Y}_3$ : the blow-up of  $\mathcal{Y}_3$  along  $\mathcal{P}_\rho$  with the exceptional divisor  $F_\rho$ .
- $\mathcal{Y}_2 \rightarrow \mathcal{Y}$ : the contraction of the exceptional divisor  $F_\rho$  to  $G_\rho \simeq \Gamma_\rho$ .
- $\mathcal{Y}_0 \rightarrow \mathcal{Y}$ : the blow-up of  $\mathcal{Y}$  along  $\mathcal{P}_\sigma \simeq \mathbb{P}(\Omega(1))$ .
- $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ : a divisorial contraction (see Subsection 6.9).

We use the same symbol  $\mathcal{P}_\sigma$  for the transform of  $\mathcal{P}_\sigma$  on  $\mathcal{Y}_2$  and further on  $\widetilde{\mathcal{Y}}$ .

*In what follows, we will use the following convention without mentioning at each time:*

- $L_\Sigma$ : the pull back on a variety  $\Sigma$  of  $\mathcal{O}(1)$  if there is a morphism  $\Sigma \rightarrow \mathbb{P}(V)$ .
- $M_\Sigma$ : the pull back on a variety  $\Sigma$  of  $\mathcal{O}_{\mathcal{H}}(1)$  if there is a morphism  $\Sigma \rightarrow \mathcal{H}$ .



**Fig.2. Birational geometries of  $\mathcal{Y}$ .**  $F_{\tilde{\mathcal{Y}}}$  in  $\tilde{\mathcal{Y}}$  represents the prime divisor parameterizing reducible conics on  $G(3, V)$  (see Proposition 6.9.1). We may identify  $\Gamma_\rho$  and  $G_\rho$  (see Subsection 6.8).

### 6.1. The Grassmann bundle $G(2, T(-1))$ .

Note that  $G(2, T(-1)) = \{([\bar{V}_2], [V_1]) \mid \bar{V}_2 \subset V/V_1\}$ . Therefore we have a natural morphism  $\rho_G: G(2, T(-1)) \rightarrow G(3, V)$  by sending  $([\bar{V}_2], [V_1]) \in G(2, T(-1))$  to  $[V_3] \in G(3, V)$ , where  $V_3$  is the inverse image of  $\bar{V}_2$  by the natural projection  $V \rightarrow V/V_1$ . Thus we obtain the following part of the big diagram:

$$(6.2) \quad \begin{array}{ccc} \mathbb{P}(T(-1)^{\wedge 2}) & \supset & G(2, T(-1)) \\ & \swarrow \rho_G & \searrow \pi_G \\ & G(3, V) & \mathbb{P}(V). \end{array}$$

Note that  $G(2, T(-1))$  is a divisor in  $\mathbb{P}(T(-1)^{\wedge 2})$  since its restrictions to fibers are quadric hypersurfaces. We may describe the linear equivalence class of  $G(2, T(-1))$ :

**Proposition 6.1.1.**  $G(2, T(-1)) \in |2H_{\mathbb{P}(T(-1)^{\wedge 2})} + L_{\mathbb{P}(T(-1)^{\wedge 2})}|$ .

*Proof.* Fix a point  $[V_1] \in \mathbb{P}(V)$ . The defining equation of  $G(2, V/V_1)$  in  $\mathbb{P}(\wedge^2(V/V_1))$  is the Plücker quadric, which defines a symmetric bi-linear form  $\wedge^2(V/V_1) \times \wedge^2(V/V_1) \rightarrow \wedge^4(V/V_1)$ . Since this globalizes to  $\wedge^2 T(-1) \times \wedge^2 T(-1) \rightarrow \wedge^4 T(-1) \simeq \mathcal{O}(1)$ , the defining equation of  $G(2, T(-1))$  in  $\mathbb{P}(T(-1)^{\wedge 2})$  is an element of  $H^0(\mathbb{P}(V), S^2(\Omega(1)^{\wedge 2}) \otimes \mathcal{O}(1))$ . This proves the assertion.  $\square$



### 6.2. The Grassmann bundle $\mathcal{G}_3 = G(3, T(-1)^{\wedge 2})$ .

Recall the definition of the  $G(3, 6)$ -bundle  $\mathcal{G}_3 \rightarrow \mathbb{P}(V)$  introduced in (5.1), which we have identified with  $G(3, T(-1)^{\wedge 2}) \rightarrow \mathbb{P}(V)$ .

We consider the following universal exact sequence of the Grassmann bundle  $G(3, T(-1)^{\wedge 2})$  over  $\mathbb{P}(V)$  (cf. [Ful, p.434]):

$$(6.3) \quad 0 \rightarrow \mathcal{S}^* \rightarrow \pi_{\mathcal{G}_3}^*(T(-1)^{\wedge 2}) \rightarrow \mathcal{Q} \rightarrow 0,$$

where  $\mathcal{S}$  is the dual of the relative universal subbundle of rank three and  $\mathcal{Q}$  is the relative universal quotient bundle of rank three. Taking the determinant and using  $\wedge^6(T(-1)^{\wedge 2}) = (\wedge^4 T(-1))^{\otimes 3} = \mathcal{O}(3)$ , we have

$$(6.4) \quad \det \mathcal{Q} = \det \mathcal{S} + 3L_{\mathcal{G}_3} = \det\{\mathcal{S}(L_{\mathcal{G}_3})\}.$$

Also, since  $T_{\mathcal{G}_3/\mathbb{P}(V)} = \mathcal{S} \otimes \mathcal{Q}$  (see [Ful, p.435]), we have

$$(6.5) \quad K_{\mathcal{G}_3} = -\det(\mathcal{Q} \otimes \mathcal{S}) + 5L_{\mathcal{G}_3} = -3(\det \mathcal{Q} + \det \mathcal{S}) - 5L_{\mathcal{G}_3} = -6\det \mathcal{Q} + 4L_{\mathcal{G}_3},$$

where we note  $\text{rank } \mathcal{S} = \text{rank } \mathcal{Q} = 3$  and we use (6.4) in the last equality.

### 6.3. Subvarieties $\mathcal{P}_\rho$ and $\mathcal{P}_\sigma$ of $\mathcal{G}_3$ .

In this subsection, we define two subvarieties  $\mathcal{P}_\rho$  and  $\mathcal{P}_\sigma$  of  $\mathcal{G}_3$ , which play central roles in the following arguments.

Fix a point  $[V_1] \in \mathbb{P}(V)$ . Recall that  $G(2, V/V_1)$  contains two disjoint families of planes. One of them consists of planes of the form  $P_{V_2/V_1} = \{[\mathbb{C}^2] \mid V_2/V_1 \subset \mathbb{C}^2\}$  for a one-dimensional subspace  $V_2/V_1$  of  $V/V_1$ . These planes define a subvariety  $P_\rho$  in  $G(3, \wedge^2 V/V_1)$  under the Plücker embedding. By construction,  $P_\rho \simeq \mathbb{P}(V/V_1)$ . Another family consists of planes of the form  $P_{V_4/V_1} = \{[\overline{V}_2] \mid V_4/V_1 \supset \overline{V}_2\}$  for a three-dimensional subspace  $V_4/V_1$  of  $V/V_1$ . These planes define a subvariety  $P_\sigma$  in  $G(3, \wedge^2 V/V_1)$ . We see that  $P_\sigma \simeq \mathbb{P}((V/V_1)^*)$ .

Two subvarieties  $P_\rho$  and  $P_\sigma$  are connected components of the orthogonal Grassmann  $OG(3, 6) \subset G(3, \wedge^2 V/V_1)$  with respect to the Plücker quadric. It is known that  $P_\rho$  and  $P_\sigma$  are projectively equivalent to the second Veronese variety  $v_2(\mathbb{P}^3)$  as the subvarieties of  $\mathbb{P}(\wedge^3 \wedge^2 (V/V_1))$ . This may be seen explicitly by using the coordinates given in Appendix A (see Proposition 6.4.1).

Now we define  $\mathcal{P}_\rho$  and  $\mathcal{P}_\sigma$  to be the subvarieties of  $G(3, T(-1)^{\wedge 2})$  whose fiber over  $[V_1] \in \mathbb{P}(V)$  coincides with  $P_\rho$  and  $P_\sigma$  respectively. By the descriptions of  $P_\rho$  and  $P_\sigma$ , it holds that

$$\mathcal{P}_\rho \simeq \mathbb{P}(T(-1)) \text{ and } \mathcal{P}_\sigma \simeq \mathbb{P}(\Omega_{\mathbb{P}(V)}^1(1)).$$

To emphasize  $P_\rho$  and  $P_\sigma$  are projectively equivalent to the second Veronese variety  $v_2(\mathbb{P}^3)$ , we write

$$\mathcal{P}_\rho = v_2(\mathbb{P}(T(-1))) \text{ and } \mathcal{P}_\sigma = v_2(\mathbb{P}(\Omega(1))).$$

Note that we have the following irreducible decomposition of  $\wedge^3(\wedge^2(V/V_1))$  as  $sl(V/V_1)$ -modules (see [FH, §19.1] for example):

$$(6.6) \quad \wedge^3(\wedge^2(V/V_1)) = \mathbf{S}^2(V/V_1) \oplus \mathbf{S}^2(V/V_1)^*.$$

We will call this “double spin” decomposition since the components in the r.h.s. are identified with  $V_{2\lambda_s}$  and  $V_{2\lambda_{\bar{s}}}$  as the  $so(\wedge^2 V/V_1) (\simeq sl(V/V_1))$ -modules, where  $\lambda_s$  and  $\lambda_{\bar{s}}$  represent the spinor and conjugate spinor weights, respectively (see [loc.cit.]). The following is the relative version of the decomposition (6.6):

**Proposition 6.3.1.**

$$(6.7) \quad \wedge^3 (T(-1)^{\wedge 2}) = \left( S^2(T(-1)) \otimes \mathcal{O}(1) \right) \oplus \left( S^2(T(-1))^* \otimes \mathcal{O}(2) \right).$$

*Proof.* By (6.6), we have

$$\wedge^3 (T(-1)^{\wedge 2}) = S^2(T(-1)) \otimes \mathcal{O}(a) \oplus S^2(T(-1))^* \otimes \mathcal{O}(b)$$

with some integers  $a$  and  $b$ . To determine  $a$  and  $b$ , we restricts this equality to a line  $l \subset \mathbb{P}(V)$ . Noting  $T(-1)|_l \simeq \mathcal{O}_l^{\oplus 3} \oplus \mathcal{O}_l(1)$ , we have

$$\begin{aligned} & \mathcal{O}_l(3) \oplus \mathcal{O}_l(2)^{\oplus 9} \oplus \mathcal{O}_l(1)^{\oplus 9} \oplus \mathcal{O}_l \\ &= (\mathcal{O}_l(2) \oplus \mathcal{O}_l(1)^{\oplus 3} \oplus \mathcal{O}_l^{\oplus 6}) \otimes \mathcal{O}_l(a) \oplus (\mathcal{O}_l(-2) \oplus \mathcal{O}_l(-1)^{\oplus 3} \oplus \mathcal{O}_l^{\oplus 6}) \otimes \mathcal{O}_l(b). \end{aligned}$$

Thus we have  $a = 1$  and  $b = 2$ .  $\square$

By the decomposition (6.7), one may write the Plücker embedding of  $\mathcal{Y}_3 = G(3, T(-1)^{\wedge 2})$  to  $\mathbb{P}(\wedge^3(T(-1)^{\wedge 2}))$  by

$$\mathcal{Y}_3 \subset \mathbb{P}(S^2(T(-1)) \otimes \mathcal{O}(1) \oplus S^2(\Omega(1)) \otimes \mathcal{O}(2)).$$

In this form, it is clear that the (relative) second Veronese varieties  $\mathcal{P}_\rho = v_2(\mathbb{P}(T(-1)))$  and  $\mathcal{P}_\sigma = v_2(\mathbb{P}(\Omega(1)))$  are contained, respectively, in the first and the second factor of the projective space.

We investigate the restrictions of  $\mathcal{Q}$  and  $\mathcal{S}$  to  $\mathcal{P}_\rho$  and  $\mathcal{P}_\sigma$ .

The relation (6.4) for the determinants may be made more precise on  $\mathcal{P}_\rho$  and  $\mathcal{P}_\sigma$  as follows:

**Proposition 6.3.2.**  $\mathcal{Q}|_{\mathcal{P}_\rho} \simeq \mathcal{S}(L_{\mathcal{Y}_3})|_{\mathcal{P}_\rho}$  and  $\mathcal{Q}|_{\mathcal{P}_\sigma} \simeq \mathcal{S}(L_{\mathcal{Y}_3})|_{\mathcal{P}_\sigma}$ .

*Proof.* Proofs of the both relations are similar, so we only prove the former. Take a point  $[P_{V_2/V_1}]$  of  $\mathcal{P}_\rho \subset G(3, T(-1)^{\wedge 2})$ . Let  $W_1$  and  $W_2$  be the fiber of  $\mathcal{S}^*$  and  $\mathcal{Q}$  at  $[P_{V_2/V_1}]$ , respectively. We compare the restrictions of the universal exact sequence (6.3) and its dual;

$$\begin{aligned} 0 &\rightarrow W_1 \rightarrow \wedge^2 V/V_1 \rightarrow W_2 \rightarrow 0, \\ 0 &\rightarrow (W_2)^* \rightarrow (\wedge^2 V/V_1)^* \rightarrow (W_1)^* \rightarrow 0. \end{aligned}$$

Note that  $\mathbb{P}(W_1) = P_{V_2/V_1}$ . Choose a basis  $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$  of  $V/V_1$  so that  $V_2/V_1 = \langle \bar{e}_1 \rangle$ . Then  $W_1 = \langle \bar{e}_1 \wedge \bar{e}_2, \bar{e}_1 \wedge \bar{e}_3, \bar{e}_1 \wedge \bar{e}_4 \rangle$ . By the non-degenerate pairing  $\wedge^2 V/V_1 \times \wedge^2 V/V_1 \rightarrow \wedge^4 V/V_1 \simeq \mathbb{C}$ , we may identify  $\wedge^2 V/V_1$  and  $(\wedge^2 V/V_1)^*$ . Under this identification, we see from the explicit basis of  $W_1$  that  $(W_2)^*$  coincides with  $W_1$  since an element of  $(W_2)^*$  is nothing but an element of  $(\wedge^2 V/V_1)^*$  vanishing on  $W_1$ . Since  $\wedge^4 V/V_1$  is a fiber of  $L_{\mathcal{P}_\rho}$  at  $[P_{V_2/V_1}]$ , we conclude that  $\mathcal{S}^*|_{\mathcal{P}_\rho} \simeq \mathcal{Q}^*|_{\mathcal{P}_\rho} \otimes L_{\mathcal{P}_\rho}$ .  $\square$

Set  $\pi_{\mathcal{P}_\rho} := \pi_{\mathcal{Y}_3}|_{\mathcal{P}_\rho}$ . Consider now the relative Euler sequence for the projective bundle  $\pi_{\mathcal{P}_\rho}: \mathcal{P}_\rho \simeq \mathbb{P}(T(-1)) \rightarrow \mathbb{P}(V)$ :

$$(6.8) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}(T(-1))}(-1) \rightarrow \pi_{\mathcal{P}_\rho}^* T(-1) \rightarrow \mathcal{R}_\rho \rightarrow 0,$$

where we set

$$(6.9) \quad \mathcal{R}_\rho := T_{\mathcal{P}_\rho/\mathbb{P}(V)}(-1) = T_{\mathcal{P}_\rho/\mathbb{P}(V)} \otimes \mathcal{O}_{\mathbb{P}(T(-1))}(-1).$$

Note that, by (6.8), we have

$$\pi_{\mathcal{P}_\rho*} \mathcal{R}_\rho = T(-1) \text{ and } \det \mathcal{R}_\rho = H_{\mathbb{P}(T(-1))} + L_{\mathcal{P}_\rho}.$$

**Proposition 6.3.3.**

- (1)  $\det \mathcal{Q}|_{\mathcal{P}_\rho} = 2(H_{\mathbb{P}(T(-1))} + L_{\mathcal{P}_\rho})$ .
- (2)  $\mathcal{Q}|_{\mathcal{P}_\rho} = \mathcal{R}_\rho^*(H_{\mathbb{P}(T(-1))} + L_{\mathcal{P}_\rho}) = \wedge^2 \mathcal{R}_\rho$ .
- (3)  $\mathcal{N}_\rho^* = \mathcal{S}^2 \mathcal{Q}^*(L_{\mathcal{Y}_3})|_{\mathcal{P}_\rho}$  and  $\mathcal{N}_\sigma^* = \mathcal{S}^2 \mathcal{Q}^*(L_{\mathcal{Y}_3})|_{\mathcal{P}_\sigma}$ , where  $\mathcal{N}_\rho$  and  $\mathcal{N}_\sigma$  are the normal bundles of  $\mathcal{P}_\rho$  and  $\mathcal{P}_\sigma$  in  $\mathcal{Y}_3$ , respectively.

*Proof.* (1) Since the Plücker embedding  $G(3, T(-1)^{\wedge 2}) \hookrightarrow \mathbb{P}(\wedge^3 T(-1)^{\wedge 2})$  is defined by the (relatively very ample) invertible sheaf  $\det \mathcal{S}$ , we have  $\det \mathcal{S} = H_{\mathbb{P}(\wedge^3 T(-1)^{\wedge 2})}$ . By the decomposition (6.7), the restriction of  $H_{\mathbb{P}(\wedge^3 T(-1)^{\wedge 2})}$  to  $\mathcal{P}_\rho$  is equal to  $2H_{H_{\mathbb{P}(T(-1))}} - L_{\mathcal{P}_\rho}$ . Therefore we have the relation (1) by (6.4).

(2) Recall that, by the construction of  $\mathcal{P}_\rho$ , a point of  $\mathcal{P}_\rho$  represents a plane  $P \simeq \mathbb{P}^2$  in the fiber of  $G(2, T(-1)) \rightarrow \mathbb{P}(V)$  of the form:

$$P := P_{V_2/V_1} = \{[\overline{V}_2] \mid V_2/V_1 \subset \overline{V}_2\} \subset G(2, V/V_1).$$

Moreover, the fiber of  $\mathcal{S}^*$  at the point  $[P] \in \mathcal{P}_\rho$  is the 3-dimensional subspace  $(V/V_2) \wedge (V_2/V_1)$  of  $\wedge^2 V/V_1$  (cf. Proof of Proposition 6.3.2). On the other hand, the relative Euler sequence (6.8) restricts at  $[P]$  to

$$0 \rightarrow V_2/V_1 \rightarrow V/V_1 \rightarrow V/V_2 \rightarrow 0.$$

Therefore  $\mathcal{S}^*|_{\mathcal{P}_\rho} \simeq \mathcal{R}_\rho \otimes \mathcal{O}_{\mathbb{P}(T(-1))}(-1)$  and then we have the relations (2) by Proposition 6.3.2 and  $\det \mathcal{R}_\rho = H_{\mathbb{P}(T(-1))} + L_{\mathcal{P}_\rho}$ .

(3) As in the proof of Proposition 6.1.1, the defining equation of  $G(2, T(-1))$  in  $\mathbb{P}(T(-1)^{\wedge 2})$  is given by a section of  $\mathcal{S}^2(T(-1)^{\wedge 2})^* \otimes \mathcal{O}(1)$ . Pulling this back to  $G(3, T(-1)^{\wedge 2})$  by the morphism  $G(3, T(-1)^{\wedge 2}) \rightarrow \mathbb{P}(V)$  and using the surjection  $\pi_{\mathcal{Y}_3}^*(T(-1)^{\wedge 2})^* \rightarrow \mathcal{S}$ , we obtain a section of  $(\mathcal{S}^2 \mathcal{S}) \otimes L_{\mathcal{Y}_3}$ . Let  $P_y \simeq \mathbb{P}^2$  be the plane in  $\mathbb{P}(T(-1)^{\wedge 2})$  corresponding to  $y \in \mathcal{Y}_3$ . Then the subscheme  $\mathcal{P}_\rho \sqcup \mathcal{P}_\sigma$  of  $\mathcal{Y}_3$  is exactly the locus where  $P_y \subset G(2, T(-1))$  holds. Hence this is given by the scheme of zeros of the section of  $(\mathcal{S}^2 \mathcal{S}) \otimes L_{\mathcal{Y}_3}$ , and we have

$$\mathcal{N}_\rho^* = \mathcal{S}^2 \mathcal{S}^*(-L_{\mathcal{Y}_3})|_{\mathcal{P}_\rho} \text{ and } \mathcal{N}_\sigma^* = \mathcal{S}^2 \mathcal{S}^*(-L_{\mathcal{Y}_3})|_{\mathcal{P}_\sigma}.$$

Then we have the property (3) by Proposition 6.3.2.  $\square$

**6.4. The rational map  $\mathcal{Y}_3 \dashrightarrow \mathcal{U}$  and the blow-up  $\mathcal{Y}_2 \rightarrow \mathcal{Y}_3$ .**

Here we construct the following part of the big diagram:

$$(6.10) \quad \begin{array}{ccccc} & & \mathcal{Y}_2 & & \\ & \rho_{\mathcal{Y}_2} \swarrow & \downarrow q_{\mathcal{Y}_2} & \searrow p_{\mathcal{Y}_2} & \\ \mathcal{Y}_3 & \dashrightarrow & \mathcal{U} & & \mathcal{Y} \\ \pi_{\mathcal{Y}_3} \downarrow & & \swarrow \pi_{\mathcal{U}} & & \\ \mathbb{P}(V) & & & & \end{array}$$

Noting (6.7), we consider the relative linear projection

$$\mathbb{P}(\wedge^3(T(-1)^{\wedge 2})) \dashrightarrow \mathcal{U} = \mathbb{P}(\mathcal{S}^2 \Omega(1))$$

from  $\mathbb{P}(\mathcal{S}^2(T(-1)))$ . Then we define  $\mathcal{Y}_3 = G(3, T(-1)^{\wedge 2}) \dashrightarrow \mathcal{U}$  as in (6.10) as its restriction to  $\mathcal{Y}_3$ . We describe it fiberwise. Take a point  $[V_1] \in \mathbb{P}(V)$ . By the decomposition (6.6),  $\mathbb{P}(\mathcal{S}^2(V/V_1))$  can be identified with the linear subspace of

$\mathbb{P}(\wedge^3(\wedge^2(V/V_1)))$  spanned by  $P_\rho = v_2(\mathbb{P}(V/V_1))$ . Therefore  $\mathbb{P}(\wedge^3(\wedge^2(V/V_1))) \dashrightarrow \mathbb{P}(\mathbb{S}^2(V/V_1)^*)$  is the linear projection from the linear hull of  $P_\rho$ .

**Proposition 6.4.1.** *For a point  $[V_1] \in \mathbb{P}(V)$ , the intersection  $\mathbb{P}(\mathbb{S}^2(V/V_1)) \cap \mathcal{Y}_3$  (resp.  $\mathbb{P}(\mathbb{S}^2(V/V_1)^*) \cap \mathcal{Y}_3$ ) is scheme theoretically equal to  $\mathcal{P}_\rho|_{\pi_{\mathcal{Y}_3}^{-1}([V_1])} \simeq v_2(\mathbb{P}(V/V_1))$  (resp.  $\mathcal{P}_\sigma|_{\pi_{\mathcal{Y}_3}^{-1}([V_1])} \simeq v_2(\mathbb{P}(V/V_1)^*)$ ).*

*Proof.* We prove the assertion only for  $\mathcal{P}_\rho$  since proof is similar in the other case. According to the decomposition  $\wedge^3(\wedge^2 V/V_1) = \mathbb{S}^2(V/V_1) \oplus \mathbb{S}^2(V/V_1)^*$ , we introduce homogeneous coordinates of the projective space  $\mathbb{P}(\wedge^3(\wedge^2 V/V_1))$  by  $[v_{ij}, w^{kl}]$  as in Appendix A. Then the Plücker ideal of  $G(3, \wedge^2 V/V_1)$  is generated by (A.2) in *loc.cit.* Since the intersection  $\mathbb{P}(\mathbb{S}^2(V/V_1)) \cap \mathcal{Y}_3$  is given by  $w_{kl} = 0$ , we see that  $\mathbb{P}(\mathbb{S}^2(V/V_1)) \cap \mathcal{Y}_3$  is given by  $2 \times 2$  minors of the matrix  $v = (v_{ij})$ .  $\square$

Let  $\rho_{\mathcal{Y}_2}: \mathcal{Y}_2 \rightarrow \mathcal{Y}_3$  be the blow-up of  $\mathcal{Y}_3$  along  $\mathcal{P}_\rho$ . We denote by  $\mathcal{P}_\sigma$  the transform on  $\mathcal{Y}_2$  of  $\mathcal{P}_\sigma$ . Then the rational map becomes a morphism  $q_{\mathcal{Y}_2}: \mathcal{Y}_2 \rightarrow \mathcal{U}$  by a general property of linear projection and Proposition 6.4.1. Denote by  $F_\rho$  the  $\rho_{\mathcal{Y}_2}$ -exceptional divisor (see Fig.2). Also recall  $\mathcal{U} \subset \mathbb{P}(V) \times \mathbb{P}(\mathbb{S}^2 V^*)$  with the projections

$$(6.11) \quad \mathbb{P}(V) \xleftarrow{\pi_{\mathcal{U}}} \mathcal{U} \xrightarrow{\text{pr}_2} \mathcal{H}.$$

Note that, by the remark ii) after (5.1), we may identify  $\mathcal{Y}_3$ ,  $\mathcal{Y}_2$  and  $\mathcal{Y}_0$  near the points corresponding to smooth  $\tau$ -conics on  $G(3, V)$ .

**Lemma 6.4.2.** *Let  $y$  be the point of  $\mathcal{Y}_2$  corresponding to a smooth  $\tau$ -conic  $\delta$  on  $G(3, V)$ . Then the point  $\text{pr}_2 \circ q_{\mathcal{Y}_2}(y) \in \mathcal{H}$  corresponds to the quadric spanned by planes parameterized by  $\delta$ . In particular, the induced rational map  $\mathcal{Y}_2 \dashrightarrow \mathcal{Y}$  birationally coincides with the rational map  $\mathcal{Y}_0 \dashrightarrow \mathcal{Y}$  constructed in Proposition 5.3.1.*

*Proof.* By the natural  $SL(V)$ -action, we may assume that the smooth  $\tau$ -conic  $\delta$  has the form  $q_\tau$  given in Example 5.1.2. Then the corresponding point  $y_3 \in \mathcal{Y}_3$  is given by  $y_3 = ([\mathbf{e}_5], \mathbb{P}(\overline{V}_3))$  with  $\overline{V}_3 = \langle \bar{\mathbf{e}}_1 \wedge \bar{\mathbf{e}}_3, \bar{\mathbf{e}}_1 \wedge \bar{\mathbf{e}}_4 + \bar{\mathbf{e}}_2 \wedge \bar{\mathbf{e}}_3, \bar{\mathbf{e}}_2 \wedge \bar{\mathbf{e}}_4 \rangle$ , where  $\bar{\mathbf{e}}_i$ 's are the natural bases of  $V/V_1$  and  $V_1 = \langle \mathbf{e}_5 \rangle$ . Now using the relation (A.1), we obtain the coordinate  $[v_{ij}, w_{kl}]$  representing the point  $\mathbb{P}(\overline{V}_3) \in G(3, \wedge^2(V/\langle \mathbf{e}_5 \rangle))$  as

$$[v_{ij}, w_{kl}] = \left[ \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \right].$$

We may consider this as a point of  $\mathcal{Y}_2$ . Then we have  $q_{\mathcal{Y}_2}(y) = ([\mathbf{e}_5], [w_{kl}])$  and  $\text{pr}_2 \circ q_{\mathcal{Y}_2}(y) = [A] \in \mathcal{H}$  with  $5 \times 5$  matrix  $A = \begin{pmatrix} w_{kl} & 0 \\ 0 & 0 \end{pmatrix}$ . For the quadric  $Q_A$ , it is easy to verify the claimed property.  $\square$

**Proposition 6.4.3.** *The morphism  $q: \mathcal{Y} \rightarrow \mathcal{H}$  gives the Stein factorization of the composite  $\mathcal{Y}_2 \rightarrow \mathcal{U} \rightarrow \mathcal{H}$ . In particular we have the desired morphism  $p_{\mathcal{Y}_2}: \mathcal{Y}_2 \rightarrow \mathcal{Y}$ .*

*Proof.* By the proof of Lemma 6.4.2 we have the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{Y}_2 & \overset{p_{\mathcal{Y}_2}}{\dashrightarrow} & \mathcal{Y} \\
 \searrow \text{pr}_2 \circ q_{\mathcal{Y}_2} & & \swarrow \rho_{\mathcal{Y}} \\
 & \mathcal{H} &
 \end{array}$$

where  $p_{\mathcal{Y}_2}$  birationally coincides with the rational map  $\mathcal{Y}_0 \dashrightarrow \mathcal{Y}$  constructed in Proposition 5.3.1. Let  $\mathcal{Y}_2 \rightarrow \mathcal{Y}' \rightarrow \mathcal{H}$  be the Stein factorization. Then both  $\mathcal{Y}'$  and  $\mathcal{Y}$  are the normalization of  $\mathcal{H}$  in the function field  $K(\mathcal{Y}_2) \simeq K(\mathcal{Y})$ . Therefore, by the uniqueness of the normalization, we have  $\mathcal{Y}' \simeq \mathcal{Y}$ , and hence  $p_{\mathcal{Y}_2}$  is a morphism.  $\square$

We now describe  $q_{\mathcal{Y}_2}$  by local computations as in Appendix A.

We say that a point  $u \in \mathcal{U}$  is of rank  $i$  if  $\text{pr}_2(u)$  corresponds to a quadric in  $\mathbb{P}(V)$  of rank  $i$ . In other words,  $u$  corresponds to a quadric of rank  $i$  in the fiber  $\mathbb{P}(V/V_1)$  of  $\mathcal{U} \rightarrow \mathbb{P}(V)$  over  $[V_1]$ , where  $[V_1] := \pi_{\mathcal{U}}(u)$ . We denote by  $\mathcal{U}_i$  the subset of  $\mathcal{U}$  consisting of  $u$  of rank  $i$ . Clearly,  $\mathcal{U}_i$  is  $\text{SL}(V)$ -invariant. Note that the closure of  $\mathcal{U}_3$  is the exceptional divisor of  $\mathcal{U} \rightarrow \mathcal{H}$ .

**Proposition 6.4.4.** *Let  $u$  be a point of  $\mathcal{U}$ .*

- (1) *If  $u$  is of rank 4, then  $q_{\mathcal{Y}_2}^{-1}(u)$  consists of two points.*
- (2) *If  $u$  is of rank 3, then  $q_{\mathcal{Y}_2}^{-1}(u)$  consists of one point. The closure of  $\mathcal{U}_3$  is  $q_{\mathcal{Y}_2}(F_\rho)$ .*
- (3) *If  $u$  is of rank 2, then  $q_{\mathcal{Y}_2}^{-1}(u)$  is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ . In particular,  $q_{\mathcal{Y}_2}^{-1}(\mathcal{U}_2)$  is of codimension 1 in  $\mathcal{Y}_2$  and  $\text{SL}(V)$ -invariant.*
- (4) *If  $u$  is of rank 1, then  $q_{\mathcal{Y}_2}^{-1}(u)$  is isomorphic to the weighted projective space  $\mathbb{P}(1^3, 2)$  (the cone over  $v_2(\mathbb{P}^2)$ ). We have  $q_{\mathcal{Y}_2}(\mathcal{P}_\sigma) = \mathcal{U}_1$ .*

*Proof.*  $u \in \mathcal{U}_i$  can be regarded as a quadric in  $\mathbb{P}(V/V_1)$  of rank  $i$ , where  $[V_1] := \pi_{\mathcal{U}}(u) \in \mathbb{P}(V)$ . All the claims (1)–(4) follow from the relations (I.1)–(I.5) in Appendix A. We write  $u \in \mathcal{U}$  as  $([V_1], [w])$  with  $[w] \in \mathbb{P}(\mathbb{S}^2(V/V_1))$ . First we show (2).

(2) If  $\text{rank } w = 3$ , then the inverse image in  $\mathcal{Y}_3$  is empty by (I.3). Note that the closure of  $\mathcal{U}_3$  is a prime divisor, which is the exceptional locus of  $\text{pr}_2$ . Since the blow-up  $\mathcal{Y}_3 \rightarrow \mathcal{Y}_2$  resolve the indeterminacy of  $\mathcal{Y}_3 \dashrightarrow \mathcal{U}$  and its exceptional locus is the prime divisor  $F_\rho$ ,  $q_{\mathcal{Y}_2}(F_\rho)$  is equal to the closure of  $\mathcal{U}_3$ . Since  $\text{pr}_2(\mathcal{U}_3)$  is the branch locus of  $\mathcal{Y} \rightarrow \mathcal{H}$ ,  $F_\rho$  is contained in the branch locus of  $\mathcal{Y}_2 \rightarrow \mathcal{U}$  by Proposition 6.4.3. In other words,  $q_{\mathcal{Y}_2}^{-1}(u)$  consists of one point.

(1) Assume that  $\text{rank } w = 4$ . Then, by the proof of (2), no points of  $F_\rho$  are mapped to  $w$ . Therefore we may consider that  $q_{\mathcal{Y}_2}^{-1}(u)$  is contained in  $\mathcal{Y}_3 \setminus F_\rho$ . Then, by (I.2) in Appendix A,  $q_{\mathcal{Y}_2}^{-1}(w)$  consists of  $[v, w]$ , where  $v = \pm w^{-1} \sqrt{\det w}$ . This shows the claim.

(3) Assume that  $\text{rank } w = 2$ . First we determine  $q_{\mathcal{Y}_2}^{-1}(u) \setminus F_\rho$ , which we consider contained in  $\mathcal{Y}_3 \setminus F_\rho$ . Then, by (I.2) and (I.4),  $q_{\mathcal{Y}_2}^{-1}(u) \setminus F_\rho$  consists of  $[v, w]$  such that  $\text{rank } v = 2$  and  $v \cdot w = 0$ . Without loss of generality, we may assume that  $u$  is represented by  $w = \begin{pmatrix} O_2 & O_2 \\ O_2 & w_{33} & w_{34} \\ w_{34} & w_{44} \end{pmatrix}$ , where  $O_2$  represents the  $2 \times 2$  zero matrix. As

the solutions of  $v.w = 0$ , we have  $v = \begin{pmatrix} v_{11} & v_{12} & O_2 \\ v_{12} & v_{22} & O_2 \\ O_2 & O_2 & O_2 \end{pmatrix}$ . Then, for a fixed  $[w]$  representing  $u$ , we have only one non-trivial equation for  $[v, tw]$  from the first equations of (A.2),

$$(6.12) \quad (v_{11}v_{22} - v_{12}^2) - t^2(w_{33}w_{44} - w_{34}^2) = 0 \quad (t \neq 0),$$

where  $v_{11}, v_{12}, v_{22}, t$  are homogeneous variables. The closure in  $\mathcal{Y}_3$  of the subvariety defined by (6.12) is isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$  and lifts isomorphically on  $\mathcal{Y}_2$  since it intersects with  $\mathcal{P}_\rho$  along a divisor and  $\mathcal{Y}_2 \rightarrow \mathcal{Y}_3$  is the blow-up along  $\mathcal{P}_\rho$ . Therefore  $q_{\mathcal{Y}_2}^{-1}(u) \simeq \mathbb{P}^1 \times \mathbb{P}^1$  as claimed.

Since it is easy to see  $\dim \mathcal{U}_2 = 4 + (9 - 3) = 10$ , we have  $\dim \pi_{\mathcal{Y}_2}^{-1}(\mathcal{U}_2) = 12$ . The  $SL(V)$ -invariance follows from that of  $\mathcal{U}_2$ .

(4) Assume that  $\text{rank } w = 1$ . We may proceed similarly to the proof of (3); first we determine  $q_{\mathcal{Y}_2}^{-1}(u) \setminus F_\rho$  and then take its closure. We may assume  $w = (a_k a_l)$  with some  $\mathbf{a} \in V/V_1 \setminus \{\mathbf{0}\}$ . Then, by (I.5),  $q_{\mathcal{Y}_2}^{-1}(u) \setminus F_\rho = \{[x_i x_j, t a_k a_l] \mid \mathbf{a} \cdot \mathbf{x} = 0, t \neq 0\}$  as the zero set of (A.2). From this, we see that  $q_{\mathcal{Y}_2}^{-1}(u)$  is the cone over  $v_2(\mathbb{P}^2)$  with the vertex  $[0, a_k a_l]$ .

By Proposition 6.4.1, we have  $q_{\mathcal{Y}_2}(\mathcal{P}_\sigma) = \{([V_1], [a_i a_j]) \mid [a] \in \mathbb{P}((V/V_1)^*), [V_1] \in \mathbb{P}(V)\}$ .  $\square$

We now investigate several relations of basic divisors on  $\mathcal{Y}_2$ .

Note that

$$(6.13) \quad K_{\mathcal{Y}_2} = \rho_{\mathcal{Y}_2}^* K_{\mathcal{Y}_3} + 5F_\rho$$

since  $\rho_{\mathcal{Y}_2}$  is the blow-up along a smooth subvariety of codimension 6. By (6.5), we have

$$(6.14) \quad K_{\mathcal{Y}_2} = -6\rho_{\mathcal{Y}_2}^* \det \mathcal{Q} + 4L_{\mathcal{Y}_2} + 5F_\rho.$$

We have the following corollary to Proposition 6.3.1

**Proposition 6.4.5.**

$$M_{\mathcal{Y}_2} = \rho_{\mathcal{Y}_2}^*(\det \mathcal{Q}) - L_{\mathcal{Y}_2} - F_\rho.$$

*Proof.* Note that  $\pi_{\mathcal{Y}_3*} \mathcal{O}_{\mathbb{P}(\wedge^3 T(-1)^{\wedge 2})}(1) = \wedge^3 \Omega(1)^{\wedge 2}$ , and  $\pi_{\mathcal{U}*} \mathcal{O}_{\mathbb{P}(S^2 \Omega(1))}(1) = S^2 T(-1)$ . Therefore, by the decomposition (6.7) and the construction of  $\mathcal{Y}_3 \dashrightarrow \mathcal{U}$  as a relative linear projection, we have

$$\rho_{\mathcal{Y}_2}^*(\det \mathcal{S}) - F_\rho = q_{\mathcal{Y}_2}^*(M_{\mathcal{U}} - 2L_{\mathcal{U}}) = M_{\mathcal{Y}_2} - 2L_{\mathcal{Y}_2}.$$

Then we have the assertion by (6.4).  $\square$

**Corollary 6.4.6.**  $\mathcal{Y}_2$  is a weak Fano manifold, namely,  $-K_{\mathcal{Y}_2}$  is nef and big.

*Proof.* Note that  $\det \mathcal{Q}$  is nef since  $\mathcal{Q}$  is the image of the surjection from  $\pi_{\mathcal{Y}_3}^* T(-1)^{\wedge 2}$ . By (6.14) and Proposition 6.4.5, we have  $-K_{\mathcal{Y}_2} = 5M_{\mathcal{Y}_2} + L_{\mathcal{Y}_2} + \rho_{\mathcal{Y}_2}^* \det \mathcal{Q}$ , which is clearly nef, and is also big since so is  $M_{\mathcal{Y}_2}$ .  $\square$

### 6.5. The $K_{\mathcal{Y}_3}$ -positive small contraction $\rho_{\mathcal{Y}_3}^+ : \mathcal{Y}_3 \rightarrow \overline{\mathcal{Y}}$ .

In this subsection, we construct a  $K_{\mathcal{Y}_3}$ -positive small contraction  $\rho_{\mathcal{Y}_3}^+ : \mathcal{Y}_3 \rightarrow \overline{\mathcal{Y}}$  contracting  $\mathcal{P}_\rho$  to  $G(2, V)$ . In Subsection 6.6, we will construct the (anti-)flip of  $\rho_{\mathcal{Y}_3}^+$ , which is the second step of our two ray game.

The following descriptions of  $\mathcal{P}_\rho \simeq \mathbb{P}(T(-1))$  are standard, however we present them for readers' convenience.

**Lemma 6.5.1.** *The variety  $\mathcal{P}_\rho$  has a  $\mathbb{P}^1$ -bundle structure over  $G(2, V)$ , by which  $\mathcal{P}_\rho$  can be identified with the total space of the universal family of lines in  $\mathbb{P}(V)$ . The divisor  $H_{\mathbb{P}(T(-1))} + L_{\mathcal{P}_\rho}$  is the pull-back of  $\mathcal{O}_{G(2, V)}(1)$  by the natural projection  $\mathcal{P}_\rho \rightarrow G(2, V)$ . In particular, for a fiber  $\gamma$  of  $\mathcal{P}_\rho \rightarrow G(2, V)$ , it holds that  $L_{\mathcal{P}_\rho}|_\gamma = \mathcal{O}_\gamma(1)$  and  $H_{\mathbb{P}(T(-1))}|_\gamma = \mathcal{O}_\gamma(-1)$ .*

*Proof.* It is easy to see that  $\mathcal{P}_\rho \simeq \mathbb{P}(T(-1))$  is isomorphic to the flag variety  $F(1, 2, V)$ . The projection  $p : F(1, 2, V) \rightarrow G(2, V)$  is nothing but the universal family of lines in  $\mathbb{P}(V)$ , i.e.,  $\mathbb{P}(\mathcal{F}^*) = F(1, 2, V) \simeq \mathcal{P}_\rho$ . Note that the projection  $q : F(1, 2, V) \rightarrow \mathbb{P}(V)$  coincides with the restriction  $\pi_{\mathcal{Y}_3}|_{\mathcal{P}_\rho}$ .

Consider the dual of the relative Euler sequence associated with  $p : \mathbb{P}(\mathcal{F}^*) \rightarrow G(2, V)$ :

$$(6.15) \quad 0 \rightarrow \Omega_{\mathbb{P}(\mathcal{F}^*)/G(2, V)} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F}^*)}(1) \rightarrow p^*\mathcal{F} \rightarrow \mathcal{O}_{\mathbb{P}(\mathcal{F}^*)}(1) \rightarrow 0.$$

Since  $\text{rank } \mathcal{F} = 2$  and  $\det \mathcal{F} = \mathcal{O}_{G(2, V)}(1)$ , we see that  $\Omega_{\mathbb{P}(\mathcal{F}^*)/G(2, V)} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F}^*)}(1) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{F}^*)}(-1) \otimes p^*\mathcal{O}_{G(2, V)}(1)$ . Now we show the pushforward of (6.15) is equal to the dual of the Euler sequence of  $\mathbb{P}(V)$ . First note that  $\mathcal{O}_{\mathbb{P}(\mathcal{F}^*)}(1) = q^*\mathcal{O}(1)$ . To show  $q_*p^*\mathcal{F} \simeq V^* \otimes \mathcal{O}_{\mathbb{P}(V)}$ , we consider the pull-back to  $F(1, 2, V)$  of the dual of the universal exact sequence on  $G(2, V)$ :

$$0 \rightarrow p^*\mathcal{G}^* \rightarrow V^* \otimes \mathcal{O}_{F(1, 2, V)} \rightarrow p^*\mathcal{F} \rightarrow 0.$$

Let  $\Gamma \simeq \mathbb{P}^3$  be the fiber of  $q$  over a point  $[V_1] \in \mathbb{P}(V)$ . Then  $p^*\mathcal{G}^*|_\Gamma \simeq \Omega_{\mathbb{P}^3}^1(1)$  since  $\Gamma$  can be identified with  $\mathbb{P}(V/V_1) \subset G(2, V)$ . Therefore, by the Leray spectral sequence, we have  $q_*p^*\mathcal{G}^* \simeq R^1q_*p^*\mathcal{G}^* \simeq 0$ , which implies that  $V^* \otimes \mathcal{O}_{\mathbb{P}(V)} \simeq q_*p^*\mathcal{F}$ . Thus the pushforward of (6.15) by  $q$  is the dual of the Euler sequence of  $\mathbb{P}(V)$ , hence we have  $q_*p^*\mathcal{O}_{G(2, V)}(1) \otimes \mathcal{O}(-1) \simeq \Omega(1)$ . This implies the claim  $p^*\mathcal{O}_{G(2, V)}(1) = H_{\mathbb{P}(T(-1))} + L_{\mathcal{P}_\rho} = H_{\mathbb{P}(T(-1))} + L_{\mathcal{P}_\rho}$ . The assertions left follow from this.  $\square$

To show there exists a  $K_{\mathcal{Y}_3}$ -positive small contraction inducing the  $\mathbb{P}^1$ -bundle  $\mathcal{P}_\rho \rightarrow G(2, V)$ , we need the following result:

**Lemma 6.5.2.** *Let  $\gamma$  be a fiber of  $\mathcal{P}_\rho \rightarrow G(2, V)$ . It holds that  $\mathcal{Q}|_\gamma \simeq \mathcal{R}_\rho^*|_\gamma \simeq \mathcal{O}_\gamma^{\oplus 3}$ . Moreover,  $\mathcal{R}_\rho$  is the pull-back of the universal quotient bundle  $\mathcal{G}$  on  $G(2, V)$ .*

*Proof.* By Lemma 6.5.1,  $(H_{\mathbb{P}(T(-1))} + L_{\mathcal{P}_\rho})|_\gamma = 0$ . Therefore, by Proposition 6.3.3 (2), we have  $\mathcal{Q}|_\gamma \simeq \mathcal{R}_\rho^*|_\gamma$ . The restriction of the relative Euler sequence (6.8) to  $\gamma$  is of the form:

$$0 \rightarrow \mathcal{O}_\gamma(1) \rightarrow \mathcal{O}_\gamma^{\oplus 3} \oplus \mathcal{O}_\gamma(1) \rightarrow \mathcal{R}_\rho|_\gamma \rightarrow 0.$$

Therefore we have  $\mathcal{R}_\rho|_\gamma \simeq \mathcal{O}_\gamma^{\oplus 3}$ .

Thus  $\mathcal{R}_\rho$  is the pull-back of some locally free sheaf of rank 3 on  $G(2, V)$ . To determine this, we regard  $\mathcal{P}_\rho$  as the flag variety  $F(1, 2, V)$  as in the proof of Lemma 6.5.1 and examine the fiber of  $\mathcal{R}_\rho$  at a point  $([V_1], [V_2]) \in F(1, 2, V)$ . At this point,



note that the relative Euler sequence (6.8) restricts to  $0 \rightarrow V_2/V_1 \rightarrow V/V_1 \rightarrow V/V_2 \rightarrow 0$ . Therefore the fiber of  $\mathcal{R}_\rho$  is  $V/V_2$ , which is nothing but the fiber of  $\mathcal{G}$  at  $[V_2] \in G(2, V)$ .  $\square$

**Proposition 6.5.3.** *There exists a  $K_{\mathcal{Y}_3}$ -positive small contraction  $\rho_{\mathcal{Y}_3}^+ : \mathcal{Y}_3 \rightarrow \overline{\mathcal{Y}}$  contracting  $\mathcal{P}_\rho$  to  $G(2, V)$ .*

*Proof.* By (6.3), we see that  $\det \mathcal{Q}$  is globally generated. Let  $\rho_{\mathcal{Y}_3}^+ : \mathcal{Y}_3 \rightarrow \overline{\mathcal{Y}}$  be the Stein factorization of the morphism defined by the base point free linear system  $|\det \mathcal{Q}|$ . By Lemma 6.5.2,  $\det \mathcal{Q}$  is trivial for any fiber of  $\mathcal{P}_\rho \rightarrow G(2, V)$  and  $\det \mathcal{Q}|_{\mathcal{P}_\rho} \not\sim 0$ , thus  $\rho_{\mathcal{Y}_3}^+$  induces the fibration  $\mathcal{P}_\rho \rightarrow G(2, V)$ . Since the Picard number  $\rho(\mathcal{Y}_3) = 2$ , the relative Picard number  $\rho(\mathcal{Y}_3/\overline{\mathcal{Y}}) = 1$ . For a fiber  $\gamma$  of  $\mathcal{P}_\rho \rightarrow G(2, V)$ , we have  $K_{\mathcal{Y}_3} \cdot \gamma = 4$  by (6.5) and Lemmas 6.5.1 and 6.5.2. Thus  $\rho_{\mathcal{Y}_3}^+$  is  $K_{\mathcal{Y}_3}$ -positive. Finally we show that the  $\rho_{\mathcal{Y}_3}^+$ -exceptional locus coincides with  $\mathcal{P}_\rho$ . Indeed, let  $r$  be a  $\rho_{\mathcal{Y}_3}^+$ -exceptional curve and  $r'$  a curve on  $\mathcal{Y}_2$  whose image on  $\mathcal{Y}_3$  is  $r$ . Note that  $L_{\mathcal{Y}_3} \cdot r > 0$  since  $L_{\mathcal{Y}_3} \cdot \gamma > 0$ . If  $r \not\subset \mathcal{P}_\rho$ , then we have  $M_{\mathcal{Y}_2} \cdot r' < 0$  by Proposition 6.4.5, a contradiction to that  $M_{\mathcal{Y}_2}$  is nef. Therefore  $r \subset \mathcal{P}_\rho$  and  $r$  must be a fiber of  $\mathcal{P}_\rho \rightarrow G(2, V)$ . Thus  $\rho_{\mathcal{Y}_3}^+$  is a small contraction since  $\mathcal{P}_\rho$  is not a divisor on  $\mathcal{Y}_3$ .  $\square$

### 6.6. Constructing a resolution $\widetilde{\mathcal{Y}}$ of $\mathcal{Y}$ and the flip $\mathcal{Y}_3 \dashrightarrow \widetilde{\mathcal{Y}}$ .

In this subsection, we construct a unique birational morphism  $\tilde{\rho}_{\mathcal{Y}_2} : \mathcal{Y}_2 \rightarrow \widetilde{\mathcal{Y}}$ , which factors the morphism  $\mathcal{Y}_2 \rightarrow \mathcal{Y}$  and contracts the exceptional divisor  $F_\rho$  of the blow-up  $\rho_{\mathcal{Y}_2} : \mathcal{Y}_2 \rightarrow \mathcal{Y}_3$  in another direction. Moreover, we show that  $\widetilde{\mathcal{Y}}$  is smooth, and  $\tilde{\rho}_{\mathcal{Y}_2} : \mathcal{Y}_2 \rightarrow \widetilde{\mathcal{Y}}$  is the blow-up of  $\widetilde{\mathcal{Y}}$  along the image  $G_\rho$  of  $F_\rho$ . Actually  $\mathcal{Y}_3 \dashrightarrow \widetilde{\mathcal{Y}}$  is the (anti-)flip of the contraction  $\rho_{\mathcal{Y}_3}^+ : \mathcal{Y}_3 \rightarrow \overline{\mathcal{Y}}$  and there exists a  $K_{\widetilde{\mathcal{Y}}}$ -negative small contraction  $\widetilde{\mathcal{Y}} \rightarrow \overline{\mathcal{Y}}$  contracting the image of  $F_\rho$  to  $G(2, V)$ . The (anti-)flip  $\mathcal{Y}_3 \dashrightarrow \widetilde{\mathcal{Y}}$  is the second step of our two ray game.

First we show that  $F_\rho$  has another projective bundle structure. By Proposition 6.3.3 (3),  $F_\rho \simeq \mathbb{P}(\mathcal{N}_\rho) \simeq \mathbb{P}(\mathcal{S}^2 \mathcal{Q}|_{\mathcal{P}_\rho}) \simeq \mathbb{P}(\mathcal{S}^2 \mathcal{R}_\rho^*)$ . Let

$$G_\rho := \mathbb{P}(\mathcal{S}^2 \mathcal{G}^*).$$

Then, by Lemma 6.5.2, we obtain the following diagram:

$$(6.16) \quad \begin{array}{ccc} F_\rho & \longrightarrow & G_\rho \\ \downarrow & & \downarrow \\ \mathcal{P}_\rho & \longrightarrow & G(2, V). \end{array}$$

**Proposition 6.6.1.** *There is a unique birational morphism  $\tilde{\rho}_{\mathcal{Y}_2} : \mathcal{Y}_2 \rightarrow \widetilde{\mathcal{Y}}$  which factors  $\mathcal{Y}_2 \rightarrow \mathcal{Y}$  and contracts  $F_\rho$  to  $G_\rho$ . Moreover,  $\widetilde{\mathcal{Y}}$  is smooth and  $\tilde{\rho}_{\mathcal{Y}_2}$  is the blow-up of  $\widetilde{\mathcal{Y}}$  along  $G_\rho$ .*

*Proof.* By (6.14) and Proposition 6.4.5, we have  $-K_{\mathcal{Y}_2} + F_\rho = 2\rho_{\mathcal{Y}_2}^*(\det \mathcal{Q}) + 4M_{\mathcal{Y}_2}$ , which is nef. Since  $\mathcal{Y}_2$  is a weak Fano manifold by Corollary 6.4.6, the linear system  $|m(-K_{\mathcal{Y}_2} + F_\rho)|$  ( $m \gg 0$ ) has no base point by the Kawamata-Shokurov base point

free theorem. We show that the associated contraction morphism  $\varphi$  is the desired one. Let  $\delta$  be a fiber of  $F_\rho \rightarrow G_\rho$ . Note that  $F_\rho \cdot \delta = -1$  since  $-F_\rho|_{F_\rho}$  is the tautological divisor for  $\mathcal{N}_\rho \simeq \mathbb{S}^2 \mathcal{Q}|_{\mathcal{P}_\rho}(-L_{\mathcal{P}_\rho})$  and the image of  $\delta$  on  $\mathcal{P}_\rho$  is a fiber of  $\mathcal{P}_\rho \rightarrow \mathbb{G}(2, V)$  by the diagram (6.16). This implies  $M_{\mathcal{Y}_2} \cdot \delta = 0$  by Proposition 6.4.5. Moreover, by Lemma 6.5.2,  $\tilde{\rho}_{\mathcal{Y}_2}^*(\det \mathcal{Q})$  is trivial for  $\delta$ . Therefore,  $\delta$  is contracted by the morphism  $\varphi$ . On the other hand, for a curve  $\delta'$  in a fiber of  $F_\rho \rightarrow \mathcal{P}_\rho$ , it holds that  $M_{\mathcal{Y}_2} \cdot \delta' > 0$  by Proposition 6.4.5. Thus only fibers  $\delta$  of  $F_\rho \rightarrow G_\rho$  are contracted by  $\varphi$  among irreducible curves on  $F_\rho$  since the Picard number of  $F_\rho$  is two.

Now recall the situation of Proposition 6.4.4 (2) and (3). Let  $r$  be a ruling of  $q_{\mathcal{Y}_2}^{-1}(u) \simeq \mathbb{P}^1 \times \mathbb{P}^1$  for a point  $u \in \mathcal{U}$  of rank 2. Then  $r$  is contracted to a point on  $\mathcal{Y}$  and  $F_\rho \cdot r > 0$ . Therefore  $\rho_{\mathcal{Y}_2}^*(\det \mathcal{Q}) \cdot r > 0$  by Proposition 6.4.5. Therefore  $r$  is not contracted to a point by  $\varphi$ . Since the relative Picard number  $\rho(\mathcal{Y}_2/\mathcal{Y}) = 2$ , we conclude that  $\varphi$  is the divisorial contraction contracting  $F_\rho$  to  $G_\rho$ .

Since  $K_{\mathcal{Y}_2} \cdot \delta = -1$  and  $F_\rho \rightarrow G_\rho$  is a  $\mathbb{P}^1$ -bundle, we see that  $\widetilde{\mathcal{Y}}$  is smooth and  $\tilde{\rho}_{\mathcal{Y}_2}$  is the blow-up of  $\widetilde{\mathcal{Y}}$  along  $G_\rho$ .  $\square$

In summary, we have obtained the following diagram:

$$(6.17) \quad \begin{array}{ccc} \mathcal{Y}_2 & \xrightarrow{\tilde{\rho}_{\mathcal{Y}_2}} & \widetilde{\mathcal{Y}} \\ \downarrow \rho_{\mathcal{Y}_2} & \begin{array}{c} \hookrightarrow \\ F_\rho \longrightarrow G_\rho \\ \downarrow \quad \downarrow \\ \mathcal{P}_\rho \longrightarrow \mathbb{G}(2, V) \end{array} & \downarrow \\ \mathcal{Y}_3 & \xrightarrow{\quad} & \widetilde{\mathcal{Y}} \end{array}$$

### 6.7. Locally free sheaves $\tilde{\mathcal{S}}_L$ , $\tilde{\mathcal{Q}}$ and $\tilde{\mathcal{T}}$ on $\widetilde{\mathcal{Y}}$ .

In this subsection, we construct three locally free sheaves  $\tilde{\mathcal{S}}_L$ ,  $\tilde{\mathcal{Q}}$  and  $\tilde{\mathcal{T}}$  on  $\widetilde{\mathcal{Y}}$ , which play central roles in our description of  $\mathcal{D}^b(\widetilde{\mathcal{Y}})$ .

Let  $\delta$  be a fiber of  $F_\rho \rightarrow G_\rho$ . Then  $\rho_{\mathcal{Y}_2}(\delta)$  is a fiber of the morphism  $\mathcal{P}_\rho \rightarrow \mathbb{G}(2, V)$  (cf. (6.17)). Therefore, by Proposition 6.3.2 and Lemma 6.5.2,  $\rho_{\mathcal{Y}_2}^* \mathcal{S}(L_{\mathcal{Y}_2})|_\delta \simeq \rho_{\mathcal{Y}_2}^* \mathcal{Q}|_\delta \simeq \mathcal{O}_\delta^{\oplus 3}$ , and then  $\rho_{\mathcal{Y}_2}^* \mathcal{S}(L_{\mathcal{Y}_2})$  and  $\rho_{\mathcal{Y}_2}^* \mathcal{Q}$  turns out to be the pull-backs of locally free sheaves on  $\widetilde{\mathcal{Y}}$ .

**Definition 6.7.1.** Define locally free sheaves  $\tilde{\mathcal{S}}_L$  and  $\tilde{\mathcal{Q}}$  on  $\widetilde{\mathcal{Y}}$  by the properties

$$\rho_{\mathcal{Y}_2}^* \mathcal{S}(L_{\mathcal{Y}_2}) = \tilde{\rho}_{\mathcal{Y}_2}^* \tilde{\mathcal{S}}_L \text{ and } \rho_{\mathcal{Y}_2}^* \mathcal{Q} = \tilde{\rho}_{\mathcal{Y}_2}^* \tilde{\mathcal{Q}}. \quad \square$$

Recall the Euler sequence (6.8) associated with  $\mathcal{P}_\rho \simeq \mathbb{P}(T(-1))$ . From the dual sequence, we have the natural surjection  $\pi_{\mathcal{P}_\rho}^* \Omega(1) \rightarrow \mathcal{O}_{\mathbb{P}(T(-1))}(1) \rightarrow 0$ . Pulling back the surjection by  $\rho_{\mathcal{Y}_2}|_{F_\rho} : F_\rho \rightarrow \mathcal{P}_\rho$ , we have

$$i^* \pi_{\mathcal{Y}_2}^* \Omega(1) = (\rho_{\mathcal{Y}_2}|_{F_\rho})^* \pi_{\mathcal{P}_\rho}^* \Omega(1) \xrightarrow{\pi} (\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(1) \rightarrow 0,$$

where  $\pi_{\mathcal{Y}_2} := \pi_{\mathcal{Y}_3} \circ \rho_{\mathcal{Y}_2}$  and  $i$  is the inclusion  $i : F_\rho \rightarrow \mathcal{Y}_2$ . Now consider the sheaf  $\mathcal{T}^*$  on  $\mathcal{Y}_2$  defined by the exact sequence

$$(6.18) \quad 0 \rightarrow \mathcal{T}^* \rightarrow \pi_{\mathcal{Y}_2}^* \Omega(1) \xrightarrow{\pi \circ i^*} (\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(1) \rightarrow 0.$$

Taking  $\mathcal{E}xt^\bullet(-, \mathcal{O}_{\mathcal{Y}_2})$  of this exact sequence, we see that  $\mathcal{T}^*$  is a locally free sheaf by [Ha, III, Ex 6.6].

**Lemma 6.7.2.**  $\mathcal{T}^*|_\delta = \mathcal{O}_\delta^{\oplus 4}$  for a fiber  $\delta$  of  $F_\rho \rightarrow G_\rho$ .

*Proof.* By Lemma 6.5.1 and (6.17), the image of  $\delta$  on  $\mathbb{P}(V)$  is a line. Therefore  $\pi_{\mathcal{Y}_2}^* \Omega(1)|_\delta \simeq \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ . Also by Lemma 6.5.1,  $(\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(1)|_\delta \simeq \mathcal{O}_{\mathbb{P}^1}(-1)$ . Therefore, by restricting (6.18) to  $\delta$ , we obtain the exact sequence

$$\mathcal{T}^*|_\delta \rightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow 0.$$

Then we have a surjection  $\mathcal{T}^*|_\delta \rightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus 3}$ , whose kernel  $\mathcal{L}$  is an invertible sheaf. By (6.18), we have  $\det \mathcal{T}^* = \mathcal{O}_{\mathcal{Y}_2}(-L_{\mathcal{Y}_2} - F_\rho)$ . Therefore, by Lemma 6.5.1 and the fact that  $F_\rho \cdot \delta = -1$ , we see that  $\det \mathcal{T}^*|_\delta = \mathcal{O}_{\mathbb{P}^1}$ , and hence  $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^1}$ . Consequently, we have  $\mathcal{T}^*|_\delta = \mathcal{O}_\delta^{\oplus 4}$ .  $\square$

Therefore  $\mathcal{T} := (\mathcal{T}^*)^*$  is the pull-back of a locally free sheaf on  $\widetilde{\mathcal{Y}}$ .

**Definition 6.7.3.** Define a locally free sheaf  $\widetilde{\mathcal{T}}$  on  $\widetilde{\mathcal{Y}}$  by the property

$$\mathcal{T} = \tilde{\rho}_{\mathcal{Y}_2}^* \widetilde{\mathcal{T}}. \quad \square$$

Later we will need the dual of the exact sequence (6.18):

**Proposition 6.7.4.**

$$(6.19) \quad 0 \rightarrow \pi_{\mathcal{Y}_2}^* T(-1) \rightarrow \mathcal{T} \rightarrow (\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(-1)(F_\rho|_{F_\rho}) \rightarrow 0.$$

*Proof.* By taking  $\mathcal{H}om(-, \mathcal{O}_{\mathcal{Y}_2})$  of (6.18), we obtain:

$$0 \rightarrow \pi_{\mathcal{Y}_2}^* T(-1) \rightarrow \mathcal{T} \rightarrow \mathcal{E}xt_{\mathcal{O}_{\mathcal{Y}_2}}^1((\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(1), \mathcal{O}_{\mathcal{Y}_2}) \rightarrow 0.$$

The claim follows from the isomorphism:

$$\mathcal{E}xt_{\mathcal{O}_{\mathcal{Y}_2}}^1((\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(1), \mathcal{O}_{\mathcal{Y}_2}) \simeq (\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(-1)(F_\rho|_{F_\rho}).$$

We derive this isomorphism by the spectral sequence

$$\begin{aligned} \mathcal{E}xt_{\mathcal{O}_{F_\rho}}^i((\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(1), \mathcal{E}xt_{\mathcal{O}_{\mathcal{Y}_2}}^j(\mathcal{O}_{F_\rho}, \mathcal{O}_{\mathcal{Y}_2})) \\ \Rightarrow \mathcal{E}xt_{\mathcal{O}_{\mathcal{Y}_2}}^{i+j}((\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(1), \mathcal{O}_{\mathcal{Y}_2}), \end{aligned}$$

and also  $\mathcal{E}xt_{\mathcal{O}_{\mathcal{Y}_2}}^1(\mathcal{O}_{F_\rho}, \mathcal{O}_{\mathcal{Y}_2}) \simeq \omega_{F_\rho} \otimes \omega_{\mathcal{Y}_2}^{-1} \simeq \mathcal{O}_{F_\rho}(F_\rho|_{F_\rho})$  and  $\mathcal{E}xt_{\mathcal{O}_{\mathcal{Y}_2}}^j(\mathcal{O}_{F_\rho}, \mathcal{O}_{\mathcal{Y}_2}) = 0$  if  $j \neq 1$ .  $\square$

### 6.8. The morphism $\mathcal{Y}_0 \rightarrow \mathcal{Y}$ and relations among $\mathcal{Y}_1$ , $\mathcal{Y}_2$ and $\widetilde{\mathcal{Y}}$ .

Here we summarize the relations among  $\mathcal{Y}_1$ ,  $\mathcal{Y}_2$ ,  $\widetilde{\mathcal{Y}}$ , and we obtain the morphism  $\mathcal{Y}_0 \rightarrow \mathcal{Y}$  which extends the birational map described in Subsection 5.3. These relations are schematically drawn in Fig.2.

Let us recall that  $\mathcal{Y}_1$  is the blow-up of  $\mathcal{Y}_3$  along  $\mathcal{P}_\rho \sqcup \mathcal{P}_\sigma$  (see Section 5), and  $\mathcal{Y}_2$  is the blow-up of  $\mathcal{Y}_3$  along  $\mathcal{P}_\rho$  (see Subsection 6.4). Therefore there exists a morphism  $\mathcal{Y}_1 \rightarrow \mathcal{Y}_2$ , which is the blow-up along  $\mathcal{P}_\sigma$ . Also we have constructed the morphism  $\tilde{\rho}_{\mathcal{Y}_2} : \mathcal{Y}_2 \rightarrow \widetilde{\mathcal{Y}}$  in Proposition 6.6.1.

Moreover, recall that the  $\rho$ -conics is parameterized by the subvariety  $\Gamma_\rho$  of the Hilbert scheme  $\mathcal{Y}_0$  of conics on  $G(3, V)$ , which is isomorphic to the  $\mathbb{P}^5$ -bundle  $\mathbb{P}(S^2\mathcal{G}^*)$  over  $G(2, V)$  (see Subsection 5.2 i). By definition,  $\Gamma_\rho$  is isomorphic to  $G_\rho = \tilde{\rho}_{\mathcal{Y}_2}(F_\rho)$  as in (6.16) and (6.17). Therefore we have a morphism  $\mathcal{Y}_0 \rightarrow \widetilde{\mathcal{Y}}$ , which is the blow-up of  $\widetilde{\mathcal{Y}}$  along  $\mathcal{P}_\sigma$  (we denote by  $\mathcal{P}_\sigma$  the image on  $\widetilde{\mathcal{Y}}$  of  $\mathcal{P}_\sigma \subset \mathcal{Y}_2$ ). It will be convenient to note the following:

**Proposition 6.8.1.** *The smooth variety  $\widetilde{\mathcal{Y}}$  parametrizes  $\tau$ - and  $\rho$ -conics on  $G(3, V)$  over  $\mathcal{Y} \setminus \mathcal{P}_\sigma$ , and the  $\sigma$ -planes in  $G(3, V)$  over  $\mathcal{P}_\sigma$ .*

*Proof.*  $\mathcal{P}_\sigma \subset \mathcal{Y}_3$  parameterizes the planes in  $G(3, V)$  containing  $\sigma$ -conics by the definition of  $\mathcal{P}_\sigma$ . Therefore we have the assertion.  $\square$

Since the morphism  $\tilde{\rho}_{\mathcal{Y}_2} : \mathcal{Y}_2 \rightarrow \widetilde{\mathcal{Y}}$  factors  $\mathcal{Y}_2 \rightarrow \mathcal{Y}$ , we obtain the induced morphism

$$\rho_{\widetilde{\mathcal{Y}}} : \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}.$$

Therefore, composed with  $\mathcal{Y}_0 \rightarrow \widetilde{\mathcal{Y}}$ , we obtain the morphism  $\mathcal{Y}_0 \rightarrow \mathcal{Y}$ . By Proposition 6.4.2, this extends the rational map  $\mathcal{Y}_0 \dashrightarrow \mathcal{Y}$  constructed in Proposition 5.3.1.

By Propositions 4.2.2 and 5.3.1, the image  $\overline{G}_\rho$  of  $G_\rho$  by  $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  is the ramification locus of  $\mathcal{Y} \rightarrow \mathcal{H}$ . Thus we have the inclusion;

$$\overline{G}_\rho \supset G_{\mathcal{Y}} \supset G_{\mathcal{Y}}^1.$$

### 6.9. Descriptions of the contraction $\rho_{\widetilde{\mathcal{Y}}} : \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ .

In this subsection, we show the contraction  $\rho_{\widetilde{\mathcal{Y}}} : \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  is a  $K_{\widetilde{\mathcal{Y}}}$ -negative extremal divisorial contraction. This is the final step of our two ray game.

The results in this subsection will be further studied in Section 7 and will be used for the computations of certain cohomology groups of locally free sheaves on  $\widetilde{\mathcal{Y}}$  in Lemma 8.1.3. In Propositions 6.9.2 and 7.3.1 below, we observe that  $\rho_{\widetilde{\mathcal{Y}}} : \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  is an interesting example of extremal divisorial contractions.

Let us first describe the exceptional locus of  $\rho_{\widetilde{\mathcal{Y}}}$ . Recall that the morphism  $\mathcal{Y}_0 \rightarrow \widetilde{\mathcal{Y}}$  is the contraction of the divisor  $\Gamma_\sigma$  (see the preceding subsection and also Fig.2). Recall also that  $G_{\mathcal{Y}}$  is the inverse image in  $\mathcal{Y}$  of the locus in  $\mathcal{H}$  of quadrics of rank less than or equal to two (see Definition 4.2.3). As we have seen in Subsection 5.2, the locus  $\Gamma_\tau$  of  $\tau$ -conics in  $\mathcal{Y}_0$  consists three different  $SL(V)$ -orbits  $\Gamma_\tau^{(k)}$  depending on the rank  $k = 1, 2, 3$ . We denote the orbit closures by  $\overline{\Gamma}_\tau^{(k)}$ .

**Proposition 6.9.1.** *The exceptional locus of  $\rho_{\widetilde{\mathcal{Y}}}$  is an  $\mathrm{SL}(V)$ -invariant prime divisor. This divisor will be denoted by  $F_{\widetilde{\mathcal{Y}}}$ . The divisor  $F_{\widetilde{\mathcal{Y}}}$  is the image of  $\overline{\Gamma}_{\tau}^{(2)}$  under  $\rho_{\mathcal{Y}_0} : \mathcal{Y}_0 \rightarrow \widetilde{\mathcal{Y}}$ . The image of  $F_{\widetilde{\mathcal{Y}}}$  under  $\rho_{\widetilde{\mathcal{Y}}} : \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  is  $G_{\mathcal{Y}}$ .*

*Proof.* By construction,  $\rho_{\widetilde{\mathcal{Y}}}$  is  $\mathrm{SL}(V)$ -equivariant,  $\widetilde{\mathcal{Y}}$  is smooth and  $\rho(\widetilde{\mathcal{Y}}) = 2$ . By Proposition 4.2.4,  $\mathcal{Y}$  is a  $\mathbb{Q}$ -factorial Gorenstein Fano variety with Picard number one. Therefore  $\rho_{\widetilde{\mathcal{Y}}}$  is neither an isomorphism nor a composite of non-trivial morphisms. Also it cannot be a small contraction. Hence we have the first assertion.

From the description of the  $\mathrm{SL}(V)$ -orbits in  $\mathcal{Y}_0$  given in Subsection 5.2,  $\widetilde{\mathcal{Y}}$  has a unique  $\mathrm{SL}(V)$ -invariant divisor, which is the image of the orbit closure of  $\tau$ -conics of rank 2. Since  $F_{\widetilde{\mathcal{Y}}}$  is  $\mathrm{SL}(V)$ -invariant, this must coincide with the image of  $\overline{\Gamma}_{\tau}^{(2)}$ .

By Proposition 6.4.4 (3) and the uniqueness of  $\mathrm{SL}(V)$ -invariant divisor on  $\widetilde{\mathcal{Y}}$ , the image of the closure of  $q_{\mathcal{Y}_2}^{-1}(\mathcal{U}_2)$  under  $\mathcal{Y}_2 \rightarrow \widetilde{\mathcal{Y}}$  must coincide with  $F_{\widetilde{\mathcal{Y}}}$ . Also the image on  $\mathcal{H}$  of the closure of  $\mathcal{U}_2$  is given by  $\mathrm{S}^2(\mathbb{P}(V^*)) \simeq G_{\mathcal{Y}}$ . Hence we see that the image of  $F_{\widetilde{\mathcal{Y}}}$  under  $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$  is  $G_{\mathcal{Y}}$ .  $\square$

The rest of this subsection is devoted to describe general fibers of  $F_{\widetilde{\mathcal{Y}}} \rightarrow G_{\mathcal{Y}}$ .

**Proposition 6.9.2.**  *$F_{\widetilde{\mathcal{Y}}}$  is generically a  $\mathbb{P}^2 \times \mathbb{P}^2$ -fibration over  $G_{\mathcal{Y}}$ . More precisely, the following holds:*

- (1) *The fiber of  $F_{\widetilde{\mathcal{Y}}} \rightarrow G_{\mathcal{Y}}$  over the point of  $G_{\mathcal{Y}}$  corresponding to a quadric of rank two is isomorphic to  $\mathbb{P}^2 \times \mathbb{P}^2$ , and its intersection with  $G_{\rho}$  is the diagonal of  $\mathbb{P}^2 \times \mathbb{P}^2$ , and*
- (2)  *$K_{\widetilde{\mathcal{Y}}} = \rho_{\widetilde{\mathcal{Y}}}^* K_{\mathcal{Y}} + 2F_{\widetilde{\mathcal{Y}}}$ . In particular,  $\rho_{\widetilde{\mathcal{Y}}}$  is a  $K_{\widetilde{\mathcal{Y}}}$ -negative divisorial extremal contraction and then  $\mathcal{Y}$  has only terminal singularities and  $\mathrm{Sing} \mathcal{Y} = G_{\mathcal{Y}}$ .*

*Proof.* From now on in the proof, we fix the point  $y$  of  $G_{\mathcal{Y}}$  corresponding to a quadric  $Q$  in  $\mathbb{P}(V)$  of rank two. Let  $\Gamma$  be the fiber of  $F_{\widetilde{\mathcal{Y}}} \rightarrow G_{\mathcal{Y}}$  over  $y$ . The quadric  $Q$  is the union of distinct 3-planes  $\mathbb{P}(V_4^{(1)})$  and  $\mathbb{P}(V_4^{(2)})$ . Set  $V_3 := V_4^{(1)} \cap V_4^{(2)}$ .

We show (1). By Proposition 6.4.4 (4), we first note that  $\Gamma$  is disjoint from (the strict transform of)  $\mathcal{P}_{\sigma}$ . Therefore  $\Gamma$  parameterizes  $\tau$ - or  $\rho$ -conics by Proposition 6.8.1. A conic  $q \subset G(3, V)$  which parametrizes planes in  $Q$  is a line pair  $l_1 \cup l_2$ . The lines  $l_i$  ( $i = 1, 2$ ) are of the form:

$$l_i = \{[\Pi] \in G(3, V) \mid V_2^{(i)} \subset \Pi \subset V_4^{(i)}\} \quad (i = 1, 2).$$

Since  $l_1 \cap l_2 \neq \emptyset$ , both  $V_2^{(1)}$  and  $V_2^{(2)}$  are 2-dimensional subspaces of  $V_3$ . Therefore such  $q$ 's are parameterized by pairs of 2-dimensional subspaces  $V_2^{(1)}$  and  $V_2^{(2)}$  of  $V_3$ , and then the fiber  $\Gamma$  is isomorphic to  $\mathbb{P}(V_3^*) \times \mathbb{P}(V_3^*) \simeq \mathbb{P}^2 \times \mathbb{P}^2$ . Note that  $l_1 \cup l_2$  is a  $\rho$ -conic if and only if  $V_2^{(1)} = V_2^{(2)}$ . Therefore, since  $G_{\rho}$  can be identified with the locus  $\Gamma_{\rho}$  of  $\rho$ -conics, we see that the intersection  $\Gamma \cap G_{\rho}$  is the diagonal of  $\mathbb{P}^2 \times \mathbb{P}^2$ .

We show (2). Let  $r$  be a line in a ruling of  $\Gamma \simeq \mathbb{P}^2 \times \mathbb{P}^2$ . It is enough to show  $K_{\widetilde{\mathcal{Y}}} \cdot r = -2$ . Indeed, assume that we prove this. By the adjunction formula

$$K_{\Gamma} = K_{F_{\widetilde{\mathcal{Y}}}}|_{\Gamma} = (K_{\widetilde{\mathcal{Y}}} + F_{\widetilde{\mathcal{Y}}})|_{\Gamma},$$

it holds  $(K_{\widetilde{\mathcal{Y}}} + F_{\widetilde{\mathcal{Y}}}) \cdot r = -3$ . Hence we have  $F_{\widetilde{\mathcal{Y}}} \cdot r = -1$ . Define a rational number  $a$  by the formula  $K_{\widetilde{\mathcal{Y}}} = \rho_{\widetilde{\mathcal{Y}}}^* K_{\mathcal{Y}} + aF_{\widetilde{\mathcal{Y}}}$ . Then it holds that  $K_{\widetilde{\mathcal{Y}}} \cdot r = aF_{\widetilde{\mathcal{Y}}} \cdot r$ . Therefore we have  $a = 2$ .

To show  $K_{\widetilde{\mathcal{Y}}} \cdot r = -2$ , we choose  $r$  such that it intersects the diagonal of  $\Gamma$ . Let  $r'$  be the strict transform on  $\mathcal{Y}_2$  of  $r$  and  $r''$  the image of  $r'$  on  $\mathcal{Y}_3$ . Since  $\mathcal{Y}_2 \rightarrow \widetilde{\mathcal{Y}}$  is the blow-up along  $G_\rho$ , we have

$$K_{\mathcal{Y}_2} \cdot r' = K_{\widetilde{\mathcal{Y}}} \cdot r + 1.$$

Moreover, by the formula (6.13), we have

$$K_{\mathcal{Y}_2} \cdot r' = K_{\mathcal{Y}_3} \cdot r'' + 5.$$

Therefore we have only to show  $K_{\mathcal{Y}_3} \cdot r'' = -6$ . The strict transform of  $\Gamma$  on  $\mathcal{Y}_2$  has a natural  $\mathbb{P}^1 \times \mathbb{P}^1$ -fibration over  $\mathbb{P}^2$  since it is the blow-up of  $\Gamma$  along the diagonal. Its fiber is described as in Proposition 6.4.4 (3) and then  $r''$  is a line in a fiber of  $\mathcal{Y}_3 \rightarrow \mathbb{P}(V)$ . Therefore we have  $K_{\mathcal{Y}_3} \cdot r'' = -6$  as desired.

Finally,  $\mathcal{Y}$  is singular along  $G_{\mathcal{Y}}$  since the minimal discrepancy for a smooth point of  $\mathcal{Y}$  is equal to  $\dim \mathcal{Y} - 1$ .  $\square$

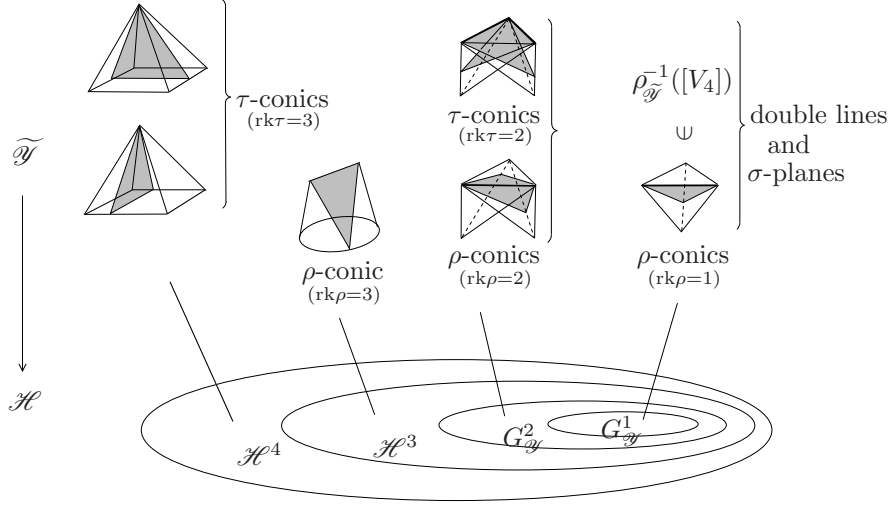
*Remark.* As a summary, we obtain the following Sarkisov link:

$$(6.20) \quad \begin{array}{ccccc} \mathcal{P}_\rho \subset \mathcal{Y}_3 & \xrightarrow{\text{(anti-)flip}} & \widetilde{\mathcal{Y}} & \supset & G_\rho \\ \downarrow \text{G(3,6)-bundle} & \searrow & \swarrow & \downarrow \text{div. cont.} & \\ \mathbb{P}(V) & & \widetilde{\mathcal{Y}} & & \mathcal{Y} \end{array}$$

$\widetilde{\mathcal{Y}}$  is a Fano manifold since  $\rho(\widetilde{\mathcal{Y}}) = 2$  and it has two  $K_{\widetilde{\mathcal{Y}}}$ -negative contraction.  $\square$

**Corollary 6.9.3.** *Let  $\mathcal{Y}_2 \rightarrow \widetilde{\mathcal{U}} \rightarrow \mathcal{U}$  be the Stein factorization of  $\mathcal{Y}_2 \rightarrow \mathcal{U}$ .  $\mathcal{Y}_2 \rightarrow \widetilde{\mathcal{U}}$  is a  $K_{\mathcal{Y}_2}$ -negative extremal divisorial contraction whose exceptional divisor is the strict transform of  $F_{\widetilde{\mathcal{Y}}}$ . The discrepancy of  $F_{\widetilde{\mathcal{Y}}}$  is 1. In particular,  $\widetilde{\mathcal{U}}$  has only terminal singularities.*

*Proof.* By a similar argument to the proof of Proposition 6.4.3, we see that  $\widetilde{\mathcal{U}} \rightarrow \mathcal{Y} \rightarrow \mathcal{H}$  is the Stein factorization of  $\mathcal{U} \rightarrow \mathcal{H}$ . The exceptional divisors of  $\mathcal{Y}_2 \rightarrow \mathcal{Y}$  consist of  $F_\rho$  and the strict transform of  $F_{\widetilde{\mathcal{Y}}}$  since  $\mathcal{Y}_2 \rightarrow \mathcal{Y}$  is decomposed as  $\mathcal{Y}_2 \rightarrow \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ . Noting Proposition 6.4.4 (2), the image on  $\mathcal{U}$  of  $F_\rho$  is contracted by  $\mathcal{U} \rightarrow \mathcal{H}$ . Therefore, looking at  $\mathcal{Y}_2 \rightarrow \widetilde{\mathcal{U}} \rightarrow \mathcal{U} \rightarrow \mathcal{H}$ , we see that the strict transform on  $\mathcal{Y}_2$  of  $F_{\widetilde{\mathcal{Y}}}$  is contracted by  $\mathcal{Y}_2 \rightarrow \widetilde{\mathcal{U}}$ . We have already shown the  $K_{\mathcal{Y}_2}$ -negativity of  $\mathcal{Y}_2 \rightarrow \widetilde{\mathcal{U}}$  in the proof of Proposition 6.9.2 (2). Indeed,  $r'$  as in *loc.cit.* is contained in a fiber of  $\mathcal{Y}_2 \rightarrow \widetilde{\mathcal{U}}$  and it satisfies  $K_{\mathcal{Y}_2} \cdot r' = -1$  as desired. Moreover, we have seen  $F_{\widetilde{\mathcal{Y}}} \cdot r = -1$  in *loc.cit.* Then the intersection number of the strict transform of  $F_{\widetilde{\mathcal{Y}}}$  and  $r'$  is also  $-1$  since the center  $G_\rho$  of  $\widetilde{\rho}_{\mathcal{Y}_2}$  is not contained in  $F_{\widetilde{\mathcal{Y}}}$  and hence the strict transform of  $F_{\widetilde{\mathcal{Y}}}$  coincides with the total transform of  $F_{\widetilde{\mathcal{Y}}}$ . Thus the discrepancy of  $F_{\widetilde{\mathcal{Y}}}$  is 1. Now we have shown all the assertions.  $\square$



**Fig.3. The fibers of the composite morphism  $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y} \rightarrow \mathcal{H}$ .**  
 $\mathcal{H}^i$  represent the loci of symmetric matrices of rank  $i$ . Note that  $\mathcal{H}^k$  may be identified with  $G_{\mathcal{Y}}^k$  for  $k = 1, 2$ .

At this stage, it is worthwhile remarking that  $Y$  constructed in [HoTa1] coincides with  $Y$  in this paper. Indeed, by Corollary 6.9.3,  $\widetilde{\mathcal{U}}$  is Cohen-Macaulay since terminal singularities are Cohen-Macaulay. Therefore,  $\widetilde{\mathcal{U}} \rightarrow \mathcal{U}$  is flat. Since  $\widetilde{\mathcal{U}} \rightarrow \mathcal{U}$  is a double cover branched along the exceptional divisor of  $\mathcal{U} \rightarrow \mathcal{H}$ , we see that  $\widetilde{\mathcal{U}} \rightarrow \mathcal{U}$  is uniquely determined by its branch locus by [F-a, p.48 (6.11)]. Therefore, we see that the restrictions of  $\widetilde{\mathcal{U}}$  and  $\mathcal{Y}$  are equal to  $\widetilde{\mathcal{U}}$  and  $Y$  as in the proof of [HoTa1, Proposition 3.12].

## 7. GLOBAL DESCRIPTIONS OF $F_{\mathcal{Y}}$ AND SPECIAL FIBERS OF $F_{\mathcal{Y}} \rightarrow G_{\mathcal{Y}}$

This section is devoted to describing the exceptional divisor  $F_{\mathcal{Y}}$ . Finding an explicit description  $F_{\mathcal{Y}} = \widehat{F}/\mathbb{Z}_2$  in Proposition 7.2.5, we determine the fibers of the contraction  $F_{\mathcal{Y}} \rightarrow G_{\mathcal{Y}}$  over points in  $G_{\mathcal{Y}}^1$  (Proposition 7.3.1, Fig.3). We further study the geometry of  $F_{\mathcal{Y}}$  in details as it will be required for calculating certain cohomologies of the sheaves  $\widetilde{\mathcal{Q}}, \widetilde{\mathcal{S}}_L, \widetilde{\mathcal{T}}$  on  $\widetilde{\mathcal{Y}}$  in Section 8. It turns out that the fibration  $F_{\mathcal{Y}} \rightarrow G_{\mathcal{Y}}$  is not flat (Propositions 6.9.2 and 7.3.1). We find a natural flattening in Proposition 7.4.1. The properties in Lemma 7.5.1 and Lemma 7.5.2 are derived for our proof of Proposition 8.1.3.

### 7.1. Double lines supported on a line.

In this subsection, we fix a point  $[V_4] \in \mathbb{P}(V^*) \simeq G_{\mathcal{Y}}^1$ . As the following proposition indicates, double lines on  $G(3, V)$  are relevant for the description of the fiber  $\rho_{\mathcal{Y}}^{-1}([V_4])$ .

**Proposition 7.1.1.** *The fiber  $\rho_{\mathcal{Y}}^{-1}([V_4])$  parameterizes planes containing double lines supported on lines of the form  $l_{V_2 V_4} := \{[\mathbb{C}^3] \mid V_2 \subset \mathbb{C}^3 \subset V_4\}$  with some  $[V_2] \in G(2, V)$  such that  $V_2 \subset V_4$ .*

*Proof.* By the proof of Proposition 6.9.2 (1), the points of  $\widetilde{\mathcal{Y}}$  corresponding to  $\rho$ - or  $\tau$ -conics of rank 2 are mapped to  $G_{\mathcal{Y}} \setminus G_{\mathcal{Y}}^1$  by  $\rho_{\mathcal{Y}}$ . Therefore, by Proposition 6.8.1, the fiber  $\rho_{\mathcal{Y}}^{-1}([V_4])$  parameterizes  $\rho$  or  $\tau$ -double lines supported on lines of the form  $l_{V_2 V_4}$ , or  $\sigma$ -planes in  $G(3, V)$  of the form  $P_{V_1 V_4}$ . Let  $q$  be a double line supported on the line  $l := l_{V_2 V_4}$ . Then  $q$  is uniquely determined by the plane  $\mathbb{P}_q^2$  as  $q = \mathbb{P}_q^2 \cap G(3, V)$  if  $q$  is not a  $\rho$ -conic. When  $q$  is a  $\rho$ -conic, then we have  $\mathbb{P}_q^2 = P_{V_2} = \{[\mathbb{C}^3] \mid V_2 \subset \mathbb{C}^3\}$ , and recover the line  $l_{V_2 V_4}$ . Therefore the assertion follows.  $\square$

Below we describe double lines on  $G(3, V)$ . Recall that, by Proposition 5.4.1, a double line on  $G(3, V)$  descends to a double line on  $G(2, V/V_1)$  for some  $V_1 \subset V$ . Thus we describe double lines in fibers of  $G(2, T(-1)) \rightarrow \mathbb{P}(V)$ .

Let  $G(2, V/V_1)$  be the fiber of  $G(2, T(-1)) \rightarrow \mathbb{P}(V)$  over a point  $[V_1] \in \mathbb{P}(V)$  with  $V_1 \subset V_4$ . We consider a line  $l$  in  $G(2, V/V_1)$  of the form:

$$(7.1) \quad l = l_{\overline{V}_1 \overline{V}_3} = \{[\mathbb{C}^2] \mid \overline{V}_1 \subset \mathbb{C}^2 \subset \overline{V}_3\} \subset G(2, V/V_1),$$

where we set  $\overline{V}_1 = V_2/V_1, \overline{V}_3 = V_4/V_1$  by fixing  $V_2$  satisfying  $V_1 \subset V_2 \subset V_4$ . Then we have the following description of double lines in  $G(2, V/V_1)$  supported on  $l$ :

**Proposition 7.1.2.** *Conics  $q$  in  $G(2, V/V_1)$  which are double lines supported on  $l = l_{\overline{V}_1 \overline{V}_3}$  are parametrized by  $\mathbb{P}(\overline{V}_1 \otimes (V/V_4) \oplus \wedge^2(\overline{V}_3/\overline{V}_1)) \simeq \mathbb{P}^1$ . The boundary point  $[\overline{V}_1 \otimes (V/V_4)]$  (resp.  $[\wedge^2(\overline{V}_3/\overline{V}_1)]$ ) represents a double line which is a  $\rho$ -conic (resp.  $\sigma$ -conic). Others corresponds to double lines which are  $\tau$ -conic.*

*More explicitly, choose a basis  $\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{e}_4\}$  of  $V/V_1$  so that  $\overline{V}_1 = \langle \bar{e}_1 \rangle$  and  $\overline{V}_3 = \langle \bar{e}_1, \bar{e}_2, \bar{e}_3 \rangle$ . Then the plane  $\mathbb{P}_q^2$  spanned by a double line  $q$  supported on  $l$  is equal to the plane spanned by the vectors  $\bar{e}_1 \wedge \bar{e}_2, \bar{e}_1 \wedge \bar{e}_3$  and  $a\bar{e}_1 \wedge \bar{e}_4 + b\bar{e}_2 \wedge \bar{e}_3$  with some  $a, b \in \mathbb{C}$  depending on  $q$ . The plane  $\mathbb{P}_q^2$  is a  $\rho$ -plane (resp.  $\sigma$ -plane) if and only if  $b = 0$  (resp.  $a = 0$ ).*



*Proof.* Choose a basis  $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3, \bar{\mathbf{e}}_4\}$  of  $V/V_1$  as in the statement. Take the Plücker coordinates  $x_{ij}$  ( $1 \leq i < j \leq 4$ ) associated to this basis. Then  $l$  is the  $(x_{12}, x_{13})$ -line. The defining equation of  $G(2, V/V_1)$  in  $\mathbb{P}(\wedge^2 V/V_1)$  is  $x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$ . Since the plane  $\mathbb{P}_q^2$  spanned by the conic  $q$  contains  $l$ ,  $\mathbb{P}_q^2$  is generated by the vectors  $\bar{\mathbf{e}}_1 \wedge \bar{\mathbf{e}}_2$ ,  $\bar{\mathbf{e}}_1 \wedge \bar{\mathbf{e}}_3$  and  $a\bar{\mathbf{e}}_1 \wedge \bar{\mathbf{e}}_4 + b\bar{\mathbf{e}}_2 \wedge \bar{\mathbf{e}}_3 + c\bar{\mathbf{e}}_2 \wedge \bar{\mathbf{e}}_4 + d\bar{\mathbf{e}}_3 \wedge \bar{\mathbf{e}}_4$  with some  $a, b, c, d \in \mathbb{C}$ . Then, by substituting  $(x_{14}, x_{23}, x_{24}, x_{34}) = (ax, bx, cx, dx)$  into the Plücker relation, we see that  $\mathbb{P}_q^2 \cap G(2, V/V_1)$  determine a double line supported on  $l$  if and only if  $c = d = 0$ . Furthermore we see that, when  $c = d = 0$ ,  $\mathbb{P}_q^2 \subset G(2, V/V_1)$  holds only for  $a = 0$  or  $b = 0$ . Note that, together with  $\bar{\mathbf{e}}_1 \wedge \bar{\mathbf{e}}_2$  and  $\bar{\mathbf{e}}_1 \wedge \bar{\mathbf{e}}_3$ , the vector  $\bar{\mathbf{e}}_1 \wedge \bar{\mathbf{e}}_4$  spans  $\bar{V}_1 \otimes V/V_4$  and the vector  $\bar{\mathbf{e}}_2 \wedge \bar{\mathbf{e}}_3$  spans  $\wedge^2 \bar{V}_3/\bar{V}_1$ . From the defining properties of the conics, we have the assertion.  $\square$

In Subsection 7.3, we will determine the fiber  $\rho_{\mathcal{Y}}^{-1}([V_4])$  of  $F_{\mathcal{Y}} \rightarrow G_{\mathcal{Y}}$  over the point  $[V_4]$ . Here we only calculate its dimension as an application of Proposition 7.1.2.

**Corollary 7.1.3.** *The dimension of the fiber  $\rho_{\mathcal{Y}}^{-1}([V_4])$  is six.*

*Proof.* By Proposition 7.1.2, the planes spanned by double lines supported on lines of the form  $l_{V_2 V_4}$  are parameterized in  $\mathcal{D}_3$  by a  $\mathbb{P}^1$ -bundle over  $F(1, 2, V_4)$ , which is six-dimensional. Since this locus is birational to  $\rho_{\mathcal{Y}}^{-1}([V_4])$ , we have the assertion.  $\square$

## 7.2. The double covering of $F_{\mathcal{Y}}$ and its birational geometry.

To determine the global structure of  $F_{\mathcal{Y}}$ , we study birational geometries of its double cover, which are interesting in themselves. A summary is given in the diagram (7.8).

By the arguments in Subsection 5.2 and Proposition 6.9.1,  $F_{\mathcal{Y}}$  is birationally equivalent to the  $\mathbb{Z}_2$ -quotient of the following  $\mathbb{Z}_2$ -variety:

$$(7.2) \quad \begin{aligned} F^{(1)} &:= \{([V_2^{(1)}], [V_2^{(2)}]; [V_4^{(1)}], [V_4^{(2)}]) \mid \\ &\quad V_2^{(1)}, V_2^{(2)} \subset V_4^{(1)} \cap V_4^{(2)}, \dim V_2^{(1)} \cap V_2^{(2)} \geq 1\} \\ &\subset G(2, V) \times G(2, V) \times \mathbb{P}(V^*) \times \mathbb{P}(V^*). \end{aligned}$$

There is a natural morphism  $F^{(1)}/\mathbb{Z}_2 \rightarrow \mathbb{S}^2\mathbb{P}(V^*)$ , and its fiber over a point  $([V_4^{(1)}], [V_4^{(2)}])$  with  $V_4^{(1)} \neq V_4^{(2)}$  is isomorphic to  $\mathbb{P}(V_3^*) \times \mathbb{P}(V_3^*)$ , where we set  $V_3 = V_4^{(1)} \cap V_4^{(2)}$ . Therefore  $F^{(1)}/\mathbb{Z}_2 \rightarrow \mathbb{S}^2\mathbb{P}(V^*)$  is birationally equivalent to  $F_{\mathcal{Y}} \rightarrow G_{\mathcal{Y}}$  by Proposition 6.9.2.

It turns out that  $F^{(1)}/\mathbb{Z}_2$  is not isomorphic to  $F_{\mathcal{Y}}$  but we can reconstruct  $F_{\mathcal{Y}}$  from  $F^{(1)}/\mathbb{Z}_2$  explicitly as follows. See the diagram (7.8) for summary.

First we construct a small resolution of  $F^{(1)}$ .

**Lemma 7.2.1.** *Let*

$$(7.3) \quad \begin{aligned} F^{(2)} &:= \{([V_2^{(1)}], [V_2^{(2)}]; [V_3]; [V_4^{(1)}], [V_4^{(2)}]) \mid V_2^{(1)}, V_2^{(2)} \subset V_3 \subset V_4^{(1)} \cap V_4^{(2)}\} \\ &\subset G(2, V) \times G(2, V) \times G(3, V) \times \mathbb{P}(V^*) \times \mathbb{P}(V^*) \end{aligned}$$

and

$$(7.4) \quad \widehat{G}' := \{([V_3]; [V_4^{(1)}], [V_4^{(2)}]) \mid V_3 \subset V_4^{(1)} \cap V_4^{(2)}\} \subset G(3, V) \times \mathbb{P}(V^*) \times \mathbb{P}(V^*).$$

Then, 1)  $\widehat{G}'$  is the blow-up of  $\widehat{G} := \mathbb{P}(V^*) \times \mathbb{P}(V^*)$  along the diagonal, and  $F^{(2)}$  has a  $\mathbb{P}^2 \times \mathbb{P}^2$ -fibration structure over  $\widehat{G}'$ . In particular,  $F^{(2)}$  is smooth. 2) Moreover, the natural morphism  $F^{(2)} \rightarrow F^{(1)}$  is a small resolution.

*Proof.* The first part is almost obvious. We only show that  $F^{(2)} \rightarrow F^{(1)}$  is a small resolution. Note that, since  $V_3 = V_2^{(1)} + V_2^{(2)}$  holds for  $F^{(2)}$  when  $V_2^{(1)} \neq V_2^{(2)}$ , or  $V_3 = V_4^{(1)} \cap V_4^{(2)}$  holds for  $F^{(2)}$  when  $V_4^{(1)} \neq V_4^{(2)}$ , the morphism  $F^{(2)} \rightarrow F^{(1)}$  is isomorphic outside the diagonal set

$$(7.5) \quad \Delta_{F^{(1)}} := \{([V_2], [V_2]; [V_4], [V_4]) \mid V_2 \subset V_4\} \simeq F(2, 4, V) \subset F^{(1)}.$$

The fiber of  $F^{(2)} \rightarrow F^{(1)}$  over a point  $([V_2], [V_2]; [V_4], [V_4]) \in \Delta_{F^{(1)}}$  is

$$\{([V_2], [V_2]; [V_3], [V_4], [V_4]) \mid [V_3] \in G(3, V), V_2 \subset V_3 \subset V_4\} \simeq \mathbb{P}^1.$$

Therefore the dimension of the exceptional set of  $F^{(2)} \rightarrow F^{(1)}$  is equal to  $\dim \Delta_{F^{(1)}} + 1 = 9$ , hence  $F^{(2)} \rightarrow F^{(1)}$  is small.  $\square$

We reconstruct  $F_{\widehat{\mathcal{Y}}}$  by constructing the (anti-)flip  $F^{(2)} \dashrightarrow F^{(4)}$  of  $F^{(2)} \rightarrow F^{(1)}$  (Lemma 7.2.2); contracting the strict transform on  $F^{(4)}$  of the inverse image in  $F^{(1)}$  of the diagonal of  $\widehat{G} = \mathbb{P}(V^*) \times \mathbb{P}(V^*)$  (Lemma 7.2.4); finally dividing the naturally induced  $\mathbb{Z}_2$ -action (Proposition 7.2.5).

**Lemma 7.2.2.** *Let  $\gamma$  be a non-trivial fiber of  $F^{(2)} \rightarrow F^{(1)}$  (recall that  $\gamma \simeq \mathbb{P}^1$  as in the proof of Lemma 7.2.1). It holds that  $\mathcal{N}_{\gamma/F^{(2)}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 8}$ .*

*There exists another small resolution  $F^{(4)} \rightarrow F^{(1)}$  which is isomorphic outside  $\Delta_{F^{(1)}}$  and whose nontrivial fiber is isomorphic to  $\mathbb{P}^2$ . The variety  $F^{(4)}$  is constructed by taking the blow-up  $F^{(3)} \rightarrow F^{(2)}$  along the exceptional locus of  $F^{(2)} \rightarrow F^{(1)}$  and contracting the exceptional divisor of this blow-up in the other direction.*

*Remark.* The small resolution  $F^{(4)} \rightarrow F^{(1)}$  as in the statement can be considered locally a family of the small resolution of a 4-dimensional singularity, which is studied in [Ka].  $\square$

*Proof.* The main point of the proof is to determine the singularities of  $F^{(1)}$ . For this purpose, set

$$B_{\widehat{\mathcal{Y}}} := \{([V_2^{(1)}], [V_4^{(1)}]) \mid V_2^{(1)} \subset V_4^{(1)}\} \subset G(2, V) \times \mathbb{P}(V^*)$$

and consider the projection  $F^{(1)} \rightarrow B_{\widehat{\mathcal{Y}}}$ . Let  $A_{\widehat{\mathcal{Y}}}$  be a fiber of this projection over a point  $([V_2^{(1)}], [V_4^{(1)}]) \in B_{\widehat{\mathcal{Y}}}$ . Note that  $A_{\widehat{\mathcal{Y}}}$  is contained in  $G(2, V_4^{(1)}) \times \mathbb{P}(V^*)$ . We choose a basis  $\{\mathbf{e}_1, \dots, \mathbf{e}_5\}$  of  $V$  such that  $V_2^{(1)} = \langle \mathbf{e}_1, \mathbf{e}_2 \rangle$  and  $V_4^{(1)} = \langle \mathbf{e}_1, \dots, \mathbf{e}_4 \rangle$ . Let  $x_{ij}$  ( $1 \leq i < j \leq 4$ ) be the Plücker coordinate of  $\wedge^2 V_4^{(1)}$  associated to this basis. A 2-dimensional subspace  $V_2^{(2)}$  of  $V_4^{(1)}$  such that  $\dim V_2^{(1)} \cap V_2^{(2)} \geq 1$  is spanned by two vectors of the forms  $a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2$  and  $b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3 + b_4 \mathbf{e}_4$ . The Plücker coordinates of such a  $V_2^{(2)}$  are  $x_{12} = a_1 b_2 - a_2 b_1$ ,  $x_{13} = a_1 b_3$ ,  $x_{14} = a_1 b_4$ ,  $x_{23} = a_2 b_3$ ,  $x_{24} = a_2 b_4$ , and  $x_{34} = 0$ . Denote by  $x_1, \dots, x_5$ , and  $y_1, \dots, y_5$  the coordinates of  $\mathbb{P}(V)$  and  $\mathbb{P}(V^*)$  respectively associated to the basis  $\{\mathbf{e}_1, \dots, \mathbf{e}_5\}$  and its dual basis. A 4-dimensional subspace  $V_4^{(2)}$  of  $V$  containing  $V_2^{(1)}$  is of the form  $\{c_3 x_3 + c_4 x_4 + c_5 x_5 = 0\}$ . Therefore  $V_4^{(2)}$  contains  $V_2^{(2)}$  if and only if  $c_3 b_3 + c_4 b_4 = 0$ .

From these observations, we see that

$$A_{\tilde{\mathcal{Y}}} = \{(x_{12}, \dots, x_{34}; y_1, \dots, y_5) \mid x_{34} = 0, \text{rank} \begin{pmatrix} x_{13} & x_{23} & -y_4 \\ x_{14} & x_{24} & y_3 \end{pmatrix} \leq 1\}.$$

Therefore  $A_{\tilde{\mathcal{Y}}}$  is singular only at the origin  $o$  of the chart with  $x_{12} \neq 0$  and  $y_5 \neq 0$ , and the singularity of  $A_{\tilde{\mathcal{Y}}}$  at  $o$  is isomorphic to the vertex of the cone over the Segre variety  $\mathbb{P}^1 \times \mathbb{P}^2$ .

We may consider  $F^{(1)} \rightarrow B_{\tilde{\mathcal{Y}}}$  locally as an equisingular family of the cone over the Segre variety  $\mathbb{P}^1 \times \mathbb{P}^2$ . It is well-known that the cone  $C$  over the Segre variety  $\mathbb{P}^1 \times \mathbb{P}^2$  has exactly two small resolutions  $p_1: C_1 \rightarrow C$  and  $p_2: C_2 \rightarrow C$ , where the exceptional locus  $E_1$  of  $p_1$  is a copy of  $\mathbb{P}^2$  with  $\mathcal{N}_{E_1/C_1} \simeq \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus 2}$ , and the exceptional locus  $E_2$  of  $p_2$  is a copy of  $\mathbb{P}^1$  with  $\mathcal{N}_{E_2/C_2} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$ . We can conclude that  $F^{(2)} \rightarrow F^{(1)}$  is locally a family of  $p_2: C_2 \rightarrow C$ , and then we have the assertion (cf. [Ka]).  $F^{(4)} \rightarrow F^{(1)}$  is nothing but locally a family of  $p_1: C_1 \rightarrow C$ .  $\square$

Let  $\Delta_{\mathbb{P}}$  be the diagonal of  $\hat{G} = \mathbb{P}(V^*) \times \mathbb{P}(V^*)$  and  $D^{(1)}$  the inverse image of  $\Delta_{\mathbb{P}}$  by the natural morphism  $F^{(1)} \rightarrow \mathbb{P}(V^*) \times \mathbb{P}(V^*)$ , namely,

$$D^{(1)} := \{([V_2^{(1)}], [V_2^{(2)}]; [V_4], [V_4]) \mid V_2^{(1)}, V_2^{(2)} \subset V_4, \dim V_2^{(1)} \cap V_2^{(2)} \geq 1\} \subset F^{(1)}.$$

Let

$$D^{(2)} := \{([V_2^{(1)}], [V_2^{(2)}]; [V_3], [V_4], [V_4]) \mid V_2^{(1)}, V_2^{(2)} \subset V_3 \subset V_4\} \subset F^{(2)}.$$

Then there exists a natural morphism  $D^{(2)} \rightarrow D^{(1)}$  over  $\Delta_{\mathbb{P}}$ , which is nothing but the restriction of  $F^{(2)} \rightarrow F^{(1)}$  as in Lemma 7.2.1. By the definitions of  $D^{(1)}$  and  $D^{(2)}$ , a non-trivial fiber  $\gamma$  of  $D^{(2)} \rightarrow D^{(1)}$  is  $\mathbb{P}^1$  (actually this is also a fiber of  $F^{(2)} \rightarrow F^{(1)}$ ). Moreover,  $D^{(2)}$  has a  $\mathbb{P}^2 \times \mathbb{P}^2$ -bundle structure over the flag variety  $F(3, 4, V)$ . In particular  $D^{(2)}$  is a smooth variety. Therefore  $D^{(1)}$  is a prime divisor on  $F^{(1)}$  and  $D^{(2)}$  is its strict transform on  $F^{(2)}$ .

**Lemma 7.2.3.** *Set*

$$(7.6) \quad \begin{aligned} D^{(4)} &:= \{([V_1]; [V_2^{(1)}], [V_2^{(2)}]; [V_4], [V_4]) \mid V_2^{(1)}, V_2^{(2)} \subset V_4, V_1 \subset V_2^{(1)} \cap V_2^{(2)}\} \\ &\subset \mathbb{P}(V) \times G(2, V) \times G(2, V) \times \mathbb{P}(V^*). \end{aligned}$$

Then  $D^{(4)}$  is the strict transform on  $F^{(4)}$  of  $D^{(2)}$ . Moreover, the restriction  $D^{(2)} \dashrightarrow D^{(4)}$  of the (anti-)flip  $F^{(2)} \dashrightarrow F^{(4)}$  is a family of Atiyah flops. Noting  $D^{(2)}$  (resp.  $D^{(4)}$ ) has a natural  $\mathbb{P}^2 \times \mathbb{P}^2$ -fibration structure over  $F(3, 4, V)$  (resp.  $F(1, 4, V)$ ), we obtain the following commutative diagram:

$$(7.7) \quad \begin{array}{ccccc} D^{(2)} & \xrightarrow{\text{Atiyah flop}} & D^{(4)} & & \\ \downarrow \mathbb{P}^2 \times \mathbb{P}^2\text{-fib.} & \searrow & \swarrow & \downarrow \mathbb{P}^2 \times \mathbb{P}^2\text{-fib.} & \\ F(3, 4, V) & & D^{(1)} & & F(1, 4, V) \\ & \searrow & \downarrow & \swarrow & \\ & & \Delta_{\mathbb{P}} & & \end{array}$$

*Proof.* In a similar way to the proof of Lemma 7.2.2, we can investigate  $D^{(2)} \rightarrow D^{(1)}$  and then we can show the following:

- For a non-trivial fiber  $\gamma$  of  $D^{(2)} \rightarrow D^{(1)}$ , it holds that  $\mathcal{N}_{\gamma/D^{(2)}} \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 8}$ .
- Temporarily, we denote by  $D^{(4)'}$  the strict transform on  $F^{(4)}$  of  $D^{(2)}$ . Then  $D^{(4)'} \rightarrow D^{(1)}$  is another small resolution which is isomorphic outside  $\Delta_{F^{(1)}}$  as in (7.5) and whose nontrivial fiber is isomorphic to  $\mathbb{P}^1$ . Moreover, this is the restriction of  $F^{(4)} \rightarrow F^{(1)}$ .

From the description of the normal bundle of  $\gamma$ , we see that the birational map  $D^{(2)} \dashrightarrow D^{(4)'}$  is a family of Atiyah flops.

On the other hand, by the definition of  $D^{(4)}$ , there is a natural morphism  $D^{(4)} \rightarrow D^{(1)}$ , which is a small resolution. To see  $D^{(4)} = D^{(4)'}$ , we have only to show  $D^{(2)}$  and  $D^{(4)}$  are not isomorphic over  $\Delta_{\mathbb{P}}$  since the flop is unique. In general, for a  $\mathbb{Z}_2$ -equivariant projective morphism  $X \rightarrow S$ , we denote by  $\rho^{\mathbb{Z}_2}(X/S)$  the rank of the group of numerical equivalence classes of  $\mathbb{Z}_2$ -invariant  $\mathbb{Q}$ -Cartier divisors of  $X$  over  $S$ . Note that  $\rho^{\mathbb{Z}_2}(X) = \rho(X/\mathbb{Z}_2)$  for a projective variety  $X$ . It is easy to see that  $\rho^{\mathbb{Z}_2}(D^{(2)}/\Delta_{\mathbb{P}}) = \rho^{\mathbb{Z}_2}(D^{(4)}/\Delta_{\mathbb{P}}) = 2$ . Moreover, by the definitions of  $D^{(2)}$  and  $D^{(4)}$ , there exist  $\mathbb{Z}_2$ -equivariant morphisms  $D^{(2)} \rightarrow F(3, 4, V)$  and  $D^{(4)} \rightarrow F(1, 4, V)$  over  $\Delta_{\mathbb{P}}$  besides to flopping contractions. Therefore  $D^{(2)}$  and  $D^{(4)}$  are not isomorphic over  $\Delta_{\mathbb{P}}$ .  $\square$

**Lemma 7.2.4.** *Let  $F^{(4)}$  be as in Lemma 7.2.2. Then there exists a divisorial contraction  $F^{(4)} \rightarrow \widehat{F}$  which contracts the strict transform  $D^{(4)}$  of  $D^{(1)}$  to the locus isomorphic to the flag variety  $F(1, 4, V)$ . The discrepancy of  $D^{(4)}$  is two.*

*Proof.* Let  $\Delta'_{\mathbb{P}}$  be the inverse image in  $\widehat{G}'$  of  $\Delta_{\mathbb{P}}$ . Note that  $\Delta'_{\mathbb{P}} \simeq F(3, 4, V)$ . Let  $\Gamma$  be a fiber of  $D^{(2)} \rightarrow \Delta'_{\mathbb{P}}$ , where we recall  $\Gamma \simeq \mathbb{P}^2 \times \mathbb{P}^2$ . Then, by the proof of Lemma 7.2.3,  $\Gamma$  intersects the flopping locus along the diagonal transversally. Take a line  $r \subset \mathbb{P}^2 \times \mathbb{P}^2$  which is contained in a fiber of a projection  $\Gamma \rightarrow \mathbb{P}^2$  and intersects the flopping locus. Then its strict transform  $r'$  on  $D^{(4)}$  is contracted by the morphism  $D^{(4)} \rightarrow F(1, 4, V)$ . Since  $F^{(2)} \rightarrow \widehat{G}'$  is a  $\mathbb{P}^2 \times \mathbb{P}^2$ -fibration and  $D^{(2)}$  is the pull-back of  $\Delta'_{\mathbb{P}}$ , we see that  $K_{F^{(2)}} \cdot r = -3$  and  $D^{(2)} \cdot r = 0$ . By the standard calculations of the changes of the intersection numbers by the flip, we have  $K_{F^{(4)}} \cdot r' = -3 + 1 = -2$  and  $D^{(4)} \cdot r' = 0 - 1 = -1$ . These equalities of the intersection numbers still hold for any line in a ruling of a fiber of  $D^{(4)} \rightarrow F(1, 4, V)$ .

We show  $-K_{F^{(4)}} + 2D^{(4)}$  is relatively nef over  $\widehat{G}$ . Let  $\gamma$  be a curve on  $F^{(4)}$  which is contracted to a point  $t$  on  $\widehat{G}$ . If  $t \notin \Delta_{\mathbb{P}}$ , then  $(-K_{F^{(4)}} + 2D^{(4)}) \cdot \gamma > 0$  since  $D^{(4)} \cap \gamma = \emptyset$  and  $F^{(4)} \rightarrow \widehat{G}$  is a  $\mathbb{P}^2 \times \mathbb{P}^2$  fibration outside  $\Delta_{\mathbb{P}}$ . If  $t \in \Delta_{\mathbb{P}}$  and  $\gamma$  is an exceptional curve of  $F^{(4)} \rightarrow F^{(1)}$ , then  $(-K_{F^{(4)}} + 2D^{(4)}) \cdot \gamma > 0$  since  $-K_{F^{(4)}} \cdot \gamma > 0$  and  $D^{(4)} \cdot \gamma > 0$ . In the remaining cases,  $t \in \Delta_{\mathbb{P}}$  and  $\gamma \subset D^{(4)}$ . Therefore we have only to consider the relative nefness of  $(-K_{F^{(4)}} + 2D^{(4)})|_{D^{(4)}}$  over  $\Delta_{\mathbb{P}}$ . Now we take as  $\gamma$  any line in a ruling of a fiber of  $D^{(4)} \rightarrow F(1, 4, V)$ . As we see above,  $(-K_{F^{(4)}} + 2D^{(4)}) \cdot \gamma = 0$ . Therefore  $(-K_{F^{(4)}} + 2D^{(4)})|_{D^{(4)}}$  is the pull-back of some divisor  $D_F$  on  $F(1, 4, V)$ . It suffices to show  $D_F$  is relatively nef over  $\Delta_{\mathbb{P}}$ , which is true since an exceptional curve of  $D^{(4)} \rightarrow D^{(1)}$  is positive for  $(-K_{F^{(4)}} + 2D^{(4)})|_{D^{(4)}}$  as above and is mapped to a curve on a fiber of  $F(1, 4, V) \rightarrow \Delta_{\mathbb{P}}$ . Therefore  $-K_{F^{(4)}} + 2D^{(4)}$  is relatively nef over  $\widehat{G}$ .

Moreover, by this argument, we see that  $(-K_{F^{(4)}} + 2D^{(4)})^{\perp} \cap \overline{\text{NE}}(F^{(4)}/\widehat{G})$  is generated by the numerical class of the curves on fibers of  $D^{(4)} \rightarrow F(1, 4, V)$ . In particular,  $(-K_{F^{(4)}} + 2D^{(4)})^{\perp} \cap \overline{\text{NE}}(F^{(4)}/\widehat{G}) \subset (K_{F^{(4)}})^{<0}$ . Therefore, by Mori

theory, there exists a contraction associated to this extremal face, which is nothing but the divisorial contraction contracting  $D^{(4)}$  to  $F(1, 4, V)$ .

By the equalities  $K_{F^{(4)}} \cdot r' = -2$  and  $D^{(4)} \cdot r' = -1$ , we see that the discrepancy of  $D^{(4)}$  is two.  $\square$

As we note above, the variety  $F^{(1)}$  has a natural  $\mathbb{Z}_2$ -action. Since all the morphisms constructed to obtain  $\widehat{F}$  from  $F^{(1)}$  are  $\mathbb{Z}_2$ -equivariant, the variety  $\widehat{F}$  also has a naturally induced  $\mathbb{Z}_2$ -action. We also note that

$$G'_{\mathcal{Y}} := \widehat{G}'/\mathbb{Z}_2 \simeq \text{Hilb}^2\mathbb{P}(V^*).$$

Now we can reconstruct  $F_{\widetilde{\mathcal{Y}}}$  as follows:

**Proposition 7.2.5.** *The  $\rho_{\widetilde{\mathcal{Y}}}$ -exceptional divisor  $F_{\widetilde{\mathcal{Y}}}$  is isomorphic to  $\widehat{F}/\mathbb{Z}_2$ .*

*Proof.* We compare the morphisms  $a: F_{\widetilde{\mathcal{Y}}} \rightarrow G_{\mathcal{Y}}$  and  $b: \widehat{F}/\mathbb{Z}_2 \rightarrow G_{\mathcal{Y}}$ . By [Ta, Lemma 5.5] for example, it suffices to check the following hold:

- Both  $F_{\widetilde{\mathcal{Y}}}$  and  $\widehat{F}/\mathbb{Z}_2$  are normal.
- the morphisms  $a$  and  $b$  are isomorphic to each other in codimension one.
- $-K_{F_{\widetilde{\mathcal{Y}}}}$  and  $-K_{\widehat{F}/\mathbb{Z}_2}$  are  $\mathbb{Q}$ -Cartier.
- $-K_{F_{\widetilde{\mathcal{Y}}}}$  is  $a$ -ample and  $-K_{\widehat{F}/\mathbb{Z}_2}$  is  $b$ -ample.

The variety  $F_{\widetilde{\mathcal{Y}}}$  is normal. Indeed, it satisfies the  $S_2$  condition since it is a Cartier divisor on a smooth variety. Moreover, it satisfies the  $R_1$  condition since it is a  $\mathbb{P}^2 \times \mathbb{P}^2$ -fibration outside the locus of codimension two by Corollary 7.1.3. We see that the variety  $\widehat{F}/\mathbb{Z}_2$  is normal by its explicit construction as above.

The morphisms  $a$  and  $b$  are isomorphic outside  $G_{\mathcal{Y}}^1$  by the proof of Proposition 6.9.2 (1) and the construction of  $F^{(1)}/\mathbb{Z}_2$ . Moreover, the inverse images of  $G_{\mathcal{Y}}^1$  by the morphism  $a$  has codimension two in  $F_{\widetilde{\mathcal{Y}}}$  by Corollary 7.1.3 and the inverse images of  $G_{\mathcal{Y}}^1$  by the morphism  $b$  has codimension two in  $\widehat{F}/\mathbb{Z}_2$  by the construction of  $\widehat{F}/\mathbb{Z}_2$ . Therefore the morphisms  $a$  and  $b$  are isomorphic to each other in codimension one.

The divisor  $-K_{F_{\widetilde{\mathcal{Y}}}}$  is  $\mathbb{Q}$ -Cartier since  $F_{\widetilde{\mathcal{Y}}}$  is a divisor on the smooth variety  $\widetilde{\mathcal{Y}}$ . Since the relative Picard number  $\rho(\widetilde{\mathcal{Y}}/\mathcal{Y})$  is one and  $a$  is generically a  $\mathbb{P}^2 \times \mathbb{P}^2$ -fibration, we see that  $-K_{F_{\widetilde{\mathcal{Y}}}}$  is  $a$ -ample.

We see that similar facts hold for the morphism  $b$ .

We see that  $-K_{\widehat{F}/\mathbb{Z}_2}$  is  $\mathbb{Q}$ -Cartier. Indeed, by Lemma 7.2.4, the discrepancy of  $D^{(4)}$  is two. Then, by the Kawamata-Shokurov base point free theorem,  $-K_{F^{(4)}} - 2D^{(4)}$  is the pull-back of a Cartier divisor on  $\widehat{F}$ , which turns out to be the anti-canonical divisor  $-K_{\widehat{F}}$ . Thus  $-K_{\widehat{F}/\mathbb{Z}_2}$  is  $\mathbb{Q}$ -Cartier.

To show  $-K_{\widehat{F}/\mathbb{Z}_2}$  is  $b$ -ample, it suffices to see the relative Picard number  $\rho((\widehat{F}/\mathbb{Z}_2)/G_{\mathcal{Y}})$  is one because  $b$  is generically a  $\mathbb{P}^2 \times \mathbb{P}^2$ -fibration. We compute  $\rho((\widehat{F}/\mathbb{Z}_2)/G_{\mathcal{Y}})$  using the above construction. The relative Picard number  $\rho(F^{(2)}/\widehat{G}')$  is two since  $F^{(2)} \rightarrow \widehat{G}'$  is a  $\mathbb{P}^2 \times \mathbb{P}^2$ -fibration and it is easy to see that it is a composite of two  $\mathbb{P}^2$ -fibrations. Moreover we have  $\rho^{\mathbb{Z}_2}(F^{(2)}/\widehat{G}') = 1$  since rulings in two directions of a fiber  $\mathbb{P}^2 \times \mathbb{P}^2$  of  $F^{(2)} \rightarrow \widehat{G}'$  are exchanged by the  $\mathbb{Z}_2$ -action. Therefore  $\rho^{\mathbb{Z}_2}(F^{(2)}) = 3$  since  $\rho^{\mathbb{Z}_2}(\widehat{G}') = 2$ . It holds that  $\rho^{\mathbb{Z}_2}(F^{(4)}) = 3$  since the flip preserves the Picard number and the flip is  $\mathbb{Z}_2$ -equivariant. Since a divisorial contraction

drops the Picard number at least by one, we have  $\rho^{\mathbb{Z}_2}(\widehat{F}) \leq 2$ . Now we see that  $\rho((\widehat{F}/\mathbb{Z}_2)/G_{\mathcal{Y}})$  is one since  $\rho(G_{\mathcal{Y}}) = 1$  and the morphism  $\widehat{F}/\mathbb{Z}_2 \rightarrow G_{\mathcal{Y}}$  is non-trivial. Therefore we conclude  $-K_{\widehat{F}/\mathbb{Z}_2}$  is  $b$ -ample.  $\square$

In summary, we have obtained the following diagram:

$$(7.8) \quad \begin{array}{ccccc} & & F^{(3)} & & \\ & \swarrow & & \searrow & \\ F^{(2)} & \xrightarrow{\text{(anti-)flip (Lem. 7.2.2)}} & F^{(4)} & & \\ & \searrow & \downarrow \text{div. cont. (Lem. 7.2.4)} & & \\ & & \widehat{F} & \xrightarrow{\mathbb{Z}_2\text{-quot.}} & F_{\mathcal{Y}} \\ & \swarrow \mathbb{P}^2 \times \mathbb{P}^2\text{-fib.} & \downarrow & \searrow \rho_{F_{\mathcal{Y}}} & \\ & & F^{(1)} & \xrightarrow{\mathbb{Z}_2\text{-quot.}} & G_{\mathcal{Y}} \\ & & \downarrow \text{diag. bl. up} & & \\ & & \widehat{G}' & \xrightarrow{\mathbb{Z}_2\text{-quot.}} & \widehat{G} \end{array}$$

### 7.3. Special fibers of $F_{\mathcal{Y}} \rightarrow G_{\mathcal{Y}}$ .

Throughout this subsection, we fix a point  $[V_4] \in G_{\mathcal{Y}}^1 \simeq v_2(\mathbb{P}(V^*))$ .

Using Proposition 7.2.5, we can describe the fiber  $\rho_{\mathcal{Y}}^{-1}([V_4])$  over the point  $[V_4]$ . To state the result, we introduce the following definitions:

$\mathcal{F}_{V_4}$  : the dual of the universal subbundle on  $G(2, V_4)$ .

$E_{\sigma}$  : the divisor of  $\mathbb{P}(\mathcal{O}_{G(2, V_4)} \oplus \mathcal{F}_{V_4}(1))$  associated to the injection  $\mathcal{F}_{V_4}(1) \rightarrow \mathcal{O}_{G(2, V_4)} \oplus \mathcal{F}_{V_4}(1)$ .

$s_{\rho}$  : the section of  $\mathbb{P}(\mathcal{O}_{G(2, V_4)} \oplus \mathcal{F}_{V_4}(1)) \rightarrow G(2, V_4)$  associated to the injection  $\mathcal{O}_{G(2, V_4)} \rightarrow \mathcal{O}_{G(2, V_4)} \oplus \mathcal{F}_{V_4}(1)$ .

Note that  $E_{\sigma} \simeq \mathbb{P}(\mathcal{F}_{V_4}) \simeq \mathbb{P}(\mathcal{F}_{V_4}^*)$  since  $\text{rank } \mathcal{F}_{V_4} = 2$ . Therefore  $E_{\sigma} \rightarrow G(2, V_4)$  can be considered to be the universal family of lines on  $\mathbb{P}(V_4)$ , and then there exists a natural projection  $E_{\sigma} \rightarrow \mathbb{P}(V_4)$ . Note also that  $E_{\sigma}$  and  $s_{\rho}$  are disjoint.

**Proposition 7.3.1.** *There is a birational morphism  $\mathbb{P}(\mathcal{O}_{G(2, V_4)} \oplus \mathcal{F}_{V_4}(1)) \rightarrow \rho_{\mathcal{Y}}^{-1}([V_4])$  contracting  $E_{\sigma}$  to  $\mathbb{P}(V_4) \subset \rho_{\mathcal{Y}}^{-1}([V_4])$  and  $\mathbb{P}(V_4)$  parameterizes  $\sigma$ -planes in  $G(3, V)$  of the form  $\mathbb{P}_{V_1 V_4}$  with some  $V_1 \subset V_4$ . The image in  $\rho_{\mathcal{Y}}^{-1}([V_4])$  of the section  $s_{\rho}$  parameterizes  $\rho$ -planes in  $G(3, V)$  of the form  $\mathbb{P}_{V_2}$  with some  $V_2 \subset V_4$  (see Section 5 for the definitions of  $\rho$ - and  $\sigma$ -planes).*

*Proof.* Since the fiber under consideration is contained in the branched locus of  $\widehat{F} \rightarrow F_{\mathcal{Y}}$ , we have only to determine the fiber  $\Gamma$  of  $\widehat{F} \rightarrow \widehat{G}$  over  $[V_4]$ , where we consider  $[V_4]$  is a point of the diagonal of  $\widehat{G}$ . Let  $\Gamma'$  be the restriction over  $[V_4]$  of the exceptional locus of  $F^{(4)} \rightarrow F^{(1)}$ . Then the fiber  $\Gamma$  is nothing but the image of  $\Gamma'$  under the divisorial contraction  $F^{(4)} \rightarrow \widehat{F}$ . Since the fiber of  $\Delta_{F^{(1)}} \rightarrow \widehat{G}$  over  $[V_4]$  is  $G(2, V_4)$ , the variety  $\Gamma'$  is a  $\mathbb{P}^2$ -bundle over  $G(2, V_4)$ . By the definition of  $D^{(4)}$  as in Lemma 7.2.3, we see that  $D^{(4)}|_{\Gamma'} = F(1, 2, V_4)$ , which is nothing but the total space  $\mathbb{P}(\mathcal{F}_{V_4}^*)$  of the universal family of lines in  $\mathbb{P}(V_4)$ . Therefore we may write  $\Gamma' = \mathbb{P}(\mathcal{A}^*)$ , where  $\mathcal{A}$  is the locally free sheaf of rank three on  $G(2, V_4)$  with a

surjection  $\mathcal{A} \rightarrow \mathcal{F}_{V_4}$ . Now we show the kernel of  $\mathcal{A} \rightarrow \mathcal{F}_{V_4}$  is  $\mathcal{O}_{G(2,V_4)}(2)$ . Note that the image of  $F(1, 2, V_4)$  by the divisorial contraction  $F^{(4)} \rightarrow \widehat{F}$  is  $\mathbb{P}(V_4)$ . Therefore, since the discrepancy of  $D^{(4)}$  for  $F^{(4)} \rightarrow \widehat{F}$  is two, and  $\mathcal{O}_{\mathbb{P}(\mathcal{F}_{V_4}^*)}(1)$  is the pull-back of  $\mathcal{O}_{\mathbb{P}(V_4)}(1)$ , we see that  $D^{(4)}|_{\Gamma'} = H_{\mathbb{P}(\mathcal{A}^*)} - 2L$ , where  $L$  is the pull-back of  $\mathcal{O}_{G(2,V_4)}(1)$ . Thus the kernel of  $\mathcal{A} \rightarrow \mathcal{F}_{V_4}$  is  $\mathcal{O}_{G(2,V_4)}(2)$ . Since the exact sequence  $0 \rightarrow \mathcal{O}_{G(2,V_4)}(2) \rightarrow \mathcal{A} \rightarrow \mathcal{F}_{V_4} \rightarrow 0$  splits, we have  $\mathcal{A}^* \simeq \mathcal{O}_{G(2,V_4)}(-2) \oplus \mathcal{F}_{V_4}^* \simeq (\mathcal{O}_{G(2,V_4)} \oplus \mathcal{F}_{V_4}(1)) \otimes \mathcal{O}_{G(2,V_4)}(-2)$ .

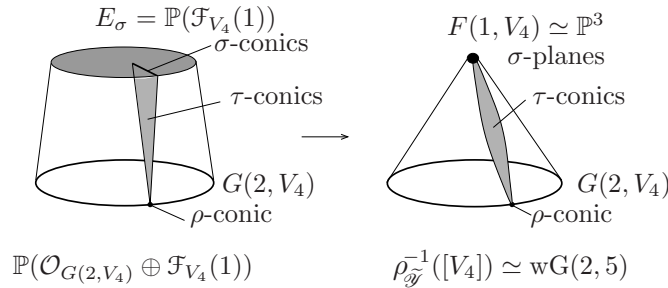
To determine the loci of  $\rho$ - and  $\sigma$ -planes, we describe the fiber of  $\mathbb{P}(\mathcal{O}_{G(2,V_4)} \oplus \mathcal{F}_{V_4}(1)) \rightarrow G(2, V_4)$  over a point  $[V_2]$  of  $G(2, V_4)$  based on Proposition 7.1.2. Fix a line  $l = l_{V_2 V_4}$ . By Proposition 7.1.2, the planes of the double lines supported on  $l$  with choices of  $V_1$  such that  $V_1 \subset V_2$  are parameterized by the following subset of  $\mathcal{Y}_3$ :

$$\Gamma_l := \{([V_1], x) \mid V_1 \subset V_2, x \in \mathbb{P}(V_2/V_1 \otimes V/V_4 \oplus \wedge^2 V_4/V_2)\}.$$

There is a natural projection from  $\Gamma_l$  to  $\mathbb{P}(V_2)$  and this gives a  $\mathbb{P}^1$ -bundle structure to  $\Gamma_l$ . Moreover, since  $V_2/V_1$  is the fiber of  $\mathcal{O}_{\mathbb{P}(V_2)}(1)$  at  $[V_1]$  and both  $V/V_4$  and  $\wedge^2 V_4/V_2$  may be regarded as the fibers of  $\mathcal{O}_{\mathbb{P}(V_2)}$  at  $[V_1]$ , the surface  $\Gamma_l$  is isomorphic to  $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1})$ . Moreover, the negative section of  $\Gamma_l$  parameterizes the planes containing  $\rho$ -double lines supported on  $l$  by Proposition 7.1.2. Therefore the strict transform  $\Gamma'_l$  on  $\mathcal{Y}_2$  of  $\Gamma_l$  is isomorphic to  $\Gamma_l$  since  $\mathcal{Y}_2 \rightarrow \mathcal{Y}_3$  is the blow-up along  $\mathcal{P}_\rho$ . Moreover, the negative section is contracted by the morphism  $\tilde{\rho}_{\mathcal{Y}_2}: \mathcal{Y}_2 \rightarrow \widetilde{\mathcal{Y}}$  since any  $\rho$ -conic has 1-parameter choices of  $V_1$  (see Section 5) and this choices are forgotten by  $\tilde{\rho}_{\mathcal{Y}_2}$  since  $\widetilde{\mathcal{Y}}$  is isomorphic to  $\mathcal{Y}_0$  outside  $\Gamma_\sigma$ . Noting  $V_2/V_1 \otimes V/V_4 \oplus \wedge^2 V_4/V_2 \simeq V_2/V_1 \otimes (V/V_4 \oplus \wedge^2 V_4/V_2 \otimes (V_2/V_1)^*)$ , we see that the forgetting morphism is nothing but

$$(7.9) \quad \Gamma'_l \rightarrow \mathbb{P}(V/V_4 \oplus \wedge^2 V_4/V_2 \otimes V_2^*) \simeq \mathbb{P}^2.$$

Therefore the image on  $\widetilde{\mathcal{Y}}$  of  $\Gamma'_l$  is  $\mathbb{P}(V/V_4 \oplus \wedge^2 V_4/V_2 \otimes V_2^*)$ , which parameterizes double lines supported on the line  $l = l_{V_2 V_4}$ . We can identify this image with the fiber of  $\mathbb{P}(\mathcal{O}_{G(2,V_4)} \oplus \mathcal{F}_{V_4}(1)) \rightarrow G(2, V_4)$  over  $[V_2]$ , where  $V/V_4$  and  $\wedge^2 V_4/V_2 \otimes V_2^*$  correspond to  $\mathcal{O}_{G(2,V_4)}$  and  $\mathcal{F}_{V_4}(1)$ , respectively. By the construction of  $\mathbb{P}(V/V_4 \oplus \wedge^2 V_4/V_2 \otimes V_2^*)$  and Proposition 7.1.2, we see that the loci in  $\mathbb{P}(\mathcal{O}_{G(2,V_4)} \oplus \mathcal{F}_{V_4}(1))$  which correspond to  $\sigma$ -double lines and  $\rho$ -double lines, respectively, are given by the divisor  $E_\sigma \simeq \mathbb{P}(\mathcal{F}_{V_4}(1))$  and the section  $s_\rho$ . Since  $\rho_{\widetilde{\mathcal{Y}}}^{-1}([V_4])$  parameterizes planes spanned by conics by Proposition 7.1.1, the claim follows.  $\square$



**Fig.4. The fiber over  $[V_4]$ .** The contraction of  $E_\sigma$  reduces  $\sigma$ -double lines to  $\sigma$ -planes.



*Remark.* (1)  $\rho_{\mathcal{Y}}^{-1}([V_4])$  is an example of the weighted Grassmann  $\text{wG}(2, 5)$  [CR, Example 2.5], which is defined in  $\mathbb{P}(1^6, 2^4)$  by the equations

$$\begin{pmatrix} 0 & x_{ij} \\ -x_{ij} & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0,$$

where  $[x_{ij}, y_k] = [x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}, y_1, \dots, y_4]$  is the (weighted) homogeneous coordinate of  $\mathbb{P}(1^6, 2^4)$ . This  $\text{wG}(2, 5)$  has singularities along  $\{x_{ij} = 0\} \simeq \mathbb{P}^3$  of type  $\frac{1}{2}(1, 1, 1)$ . Over the complement of this singular locus, we see  $\text{wG}(2, 5)$  has a  $\mathbb{C}^2$ -fibration over  $\text{G}(2, 4)$ . In Fig.4, we have depicted schematically the fiber  $\rho_{\mathcal{Y}}^{-1}([V_4])$  over  $[V_4] \in G_{\mathcal{Y}}^1$ .

(2) It is worth while comparing the descriptions of the fibers of  $\rho_{\mathcal{Y}}: F_{\mathcal{Y}} \rightarrow G_{\mathcal{Y}}$  in Propositions 6.9.2 and 7.3.1 with those of the fibers of  $q_{\mathcal{Y}_2}: \mathcal{Y}_2 \rightarrow \mathcal{U}$  in Proposition 6.4.4 (3) and (4).

(i) Suppose that a quadric  $Q = Q_A$  of rank two is given as in Proposition 6.9.2. The fiber  $\rho_{\mathcal{Y}}^{-1}([Q]) \simeq \mathbb{P}(V_3^*) \times \mathbb{P}(V_3^*)$  parameterizes reducible conics  $l = l_{V_2^{(1)}V_4^{(1)}} \cup l_{V_2^{(2)}V_4^{(2)}}$  by  $V_2^{(1)}, V_2^{(2)} \subset V_3$ . When we choose a point  $[V_1]$  in the vertex of  $Q$  and consider  $u = ([V_1], A) \in \mathcal{Y}_2$ , it is clear that the fiber  $q_{\mathcal{Y}_2}^{-1}(u) \simeq \mathbb{P}^1 \times \mathbb{P}^1$  is identified with  $\mathbb{P}((V_3/V_1)^*) \times \mathbb{P}((V_3/V_1)^*)$  which parameterizes the corresponding conics in  $\text{G}(2, V/V_1)$ . See also the proof of Proposition 6.9.2 (2).

(ii) The fiber  $\rho_{\mathcal{Y}}^{-1}([V_4]) \simeq \text{wG}(2, 5)$  has a similar correspondence to the fiber  $q_{\mathcal{Y}_2}^{-1}(u) \simeq \mathbb{P}(1^3, 2)$  over a point  $u = ([V_1], A)$  of rank 1 with  $A$  representing  $[V_4]$ , i.e.,  $\ker A = V_4$ . To see this, define a subset in  $\mathcal{Y}_3$ ,  $\Gamma(V_4) := \bigcup_{V_1 \subset V_2 \subset V_4} \Gamma_{l_{V_2V_4}}$ , where  $\Gamma_{l_{V_2V_4}}$  is defined for the line  $l_{V_2V_4}$  in the proof of Proposition 7.3.1. This has a natural  $\mathbb{P}^1$ -bundle structure  $\Gamma(V_4) \rightarrow \text{F}(1, 2, V_4)$ . Then the forgetting morphism (7.9) in the proof of Proposition 7.3.1 entails the fibration  $\mathbb{P}(\mathcal{O}_{\text{G}(2, V_4)} \oplus \mathcal{F}_{V_4}(1)) \rightarrow \text{G}(2, V_4)$ .  $\rho_{\mathcal{Y}}^{-1}([V_4])$  is obtained from  $\mathbb{P}(\mathcal{O}_{\text{G}(2, V_4)} \oplus \mathcal{F}_{V_4}(1))$  by contracting the divisor  $E_\sigma = \mathbb{P}(\mathcal{F}_{V_4}(1))$ . We may also make another fibration  $\Gamma(V_4)|_{V_1} \rightarrow \text{F}(V_1, 2, V_4) \simeq \mathbb{P}(V_4/V_1)$  by fixing a point  $[V_1] \in \mathbb{P}(V_4)$ , where  $\Gamma(V_4)|_{V_1}$  is the obvious restriction of the set  $\Gamma(V_4)$  (see the definition  $\Gamma_{l_{V_2V_4}}$ ). It is easy to see that  $\Gamma(V_4)|_{V_1} \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}(V_4/V_1)}(-1) \oplus \mathcal{O}_{\mathbb{P}(V_4/V_1)}(1))$  and also that the  $\sigma$ -conics correspond to the divisor  $\mathbb{P}(\mathcal{O}_{\mathbb{P}(V_4/V_1)}(1))$ . We note that these  $\sigma$ -conics lie on a  $\sigma$ -plane  $\mathbb{P}_{V_1V_4}$ . Contracting this divisor to a point, we obtain a cone over  $v_2(\mathbb{P}(V_4/V_1))$ , which describes the fiber  $q_{\mathcal{Y}_2}^{-1}(u) \simeq \mathbb{P}(1^3, 2)$ .  $\square$

#### 7.4. Describing a flattening of $F_{\mathcal{Y}} \rightarrow G_{\mathcal{Y}}$ .

Here we describe fibers of  $F^{(3)} \rightarrow \widehat{G}'$  as in the diagram (7.8) explicitly and show its flatness.

**Proposition 7.4.1.** (1) *The fiber of  $F^{(3)} \rightarrow \widehat{G}'$  over a point  $([V_3]; [V_4^1], [V_4^2])$  with  $V_4^1 \neq V_4^2$  is  $\mathbb{P}(V_3^*) \times \mathbb{P}(V_3^*)$ .*

(2) *The fiber of  $F^{(3)} \rightarrow \widehat{G}'$  over a point  $([V_3]; [V_4], [V_4])$  is the union of the following two 4-dimensional varieties  $A$  and  $B$ :*

- $A \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}(V_3^*)} \oplus \mathcal{T}_{\mathbb{P}(V_3^*)})$ , which is isomorphic to the restriction of  $\mathbb{P}(\mathcal{O}_{\text{G}(2, V_4)} \oplus \mathcal{F}_{V_4}(1))$  over  $\mathbb{P}(V_3^*) = \text{G}(2, V_3) \subset \text{G}(2, V_4)$ .



- $B$  is the blow-up of  $\mathbb{P}(V_3^*) \times \mathbb{P}(V_3^*)$  along the diagonal  $\Delta_{V_3}$ , hence there exists a morphism  $p_B: B \rightarrow \mathbb{P}(V_3)$  induced from the rational map

$$\begin{aligned} \mathbb{P}(V_3^*) \times \mathbb{P}(V_3^*) \setminus \Delta_{V_3} &\rightarrow \mathbb{P}(V_3) \\ ([V_2^1], [V_2^2]) &\mapsto [V_2^1 \cap V_2^2], \end{aligned}$$

and  $p_B$  is a  $\mathbb{P}^1 \times \mathbb{P}^1$ -bundle.

In particular the morphism  $F^{(3)} \rightarrow \widehat{G}'$  is flat. Moreover, the intersection  $E_{AB} := A \cap B$  is  $\mathbb{P}(T_{\mathbb{P}(V_3^*)})$  in  $A$ , which is the restriction of  $E_\sigma$ , and is the exceptional divisor of  $B \rightarrow \mathbb{P}(V_3^*) \times \mathbb{P}(V_3^*)$  in  $B$ .

*Proof.* Part (1) follows from the construction of  $F^{(2)} \rightarrow \widehat{G}'$ .

We show Part (2). The fiber of  $F^{(2)} \rightarrow \widehat{G}'$  over a point  $([V_3]; [V_4], [V_4])$  is  $\mathbb{P}(V_3^*) \times \mathbb{P}(V_3^*)$ . The intersection of the fiber  $\mathbb{P}(V_3^*) \times \mathbb{P}(V_3^*)$  with the exceptional locus of  $F^{(2)} \rightarrow F^{(1)}$  is

$$\{([V_2], [V_2]; [V_3]; [V_4], [V_4]) \mid V_2 \subset V_3\} \simeq \mathbb{P}^2,$$

which is nothing but the diagonal of  $\mathbb{P}(V_3^*) \times \mathbb{P}(V_3^*)$ . Therefore we have  $B$  as an irreducible component of the fiber of  $F^{(3)} \rightarrow \widehat{G}'$  over the point  $([V_3]; [V_4], [V_4])$ .

Another component  $A$  is a  $\mathbb{P}^2$ -bundle over the diagonal of  $\mathbb{P}(V_3^*) \times \mathbb{P}(V_3^*)$  since the exceptional divisor of  $F^{(3)} \rightarrow F^{(2)}$  is a  $\mathbb{P}^2$ -bundle over the exceptional locus of  $F^{(2)} \rightarrow F^{(1)}$ . Since the image on  $F^{(1)}$  of the diagonal  $\Delta_{V_3}$  of  $\mathbb{P}(V_3^*) \times \mathbb{P}(V_3^*)$  is equal to  $G(2, V_3) = \mathbb{P}(V_3^*)$  in  $G(2, V_4)$ , the image of  $A$  in  $F^{(4)}$  is the restriction of  $\mathbb{P}(\mathcal{O}_{G(2, V_4)} \oplus \mathcal{F}_{V_4}(1))$  over  $\mathbb{P}(2, V_3)$ . Therefore we obtain the description of  $A$  as in the statement since  $\mathcal{F}_{V_4}|_{\mathbb{P}(V_3^*)} \simeq T_{\mathbb{P}(V_3^*)}(-1)$  and  $\mathcal{N}_{\Delta_{V_3}} \cong T_{\mathbb{P}(V_3^*)}$  for the normal bundle  $\mathcal{N}_{\Delta_{V_3}}$  of the diagonal  $\Delta_{V_3}$ .

The assertion about  $A \cap B$  is easily seen.  $\square$

By Proposition 7.4.1, we obtain the flattening of  $F_{\widetilde{\mathcal{Y}}} \rightarrow G_{\mathcal{Y}}$  as follows:

$$(7.10) \quad \begin{array}{ccc} F^{(3)} & \longrightarrow & F_{\widetilde{\mathcal{Y}}} \\ \downarrow & & \downarrow \\ \widehat{G}' & \longrightarrow & G_{\mathcal{Y}}. \end{array}$$

The flatness of the morphism  $F^{(3)} \rightarrow \widehat{G}'$  is crucial for the proof of Lemma 8.1.3. By this property, we can reduce computations of cohomology groups on  $F_{\widetilde{\mathcal{Y}}}$  to those on  $F^{(3)}$  and then those on special fibers of  $F^{(3)} \rightarrow \widehat{G}'$ .

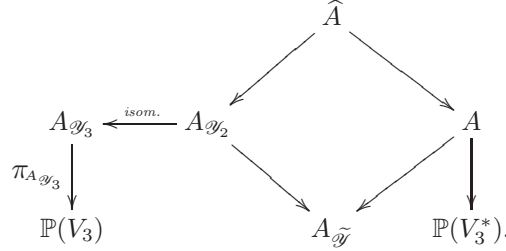
### 7.5. The pull-backs of $\widetilde{\mathcal{Q}}$ , $\widetilde{\mathcal{S}}_L$ , and $\widetilde{\mathcal{T}}$ on $A$ and $B$ .

In this subsection, we consider the situation of Proposition 7.4.1 (2). In particular, we fix  $V_3$  and  $V_4$ . We describe the pull-backs of the divisor  $F_{\widetilde{\mathcal{Y}}}$  and the sheaves  $\widetilde{\mathcal{S}}_L$ ,  $\widetilde{\mathcal{Q}}$ ,  $\widetilde{\mathcal{T}}$  on  $A$  and  $B$ . For this purpose, we study the birational geometry of  $A$  as follows:

**Lemma 7.5.1.** *Denote by  $A_{\widetilde{\mathcal{Y}}}$  the image of  $A$  on  $\widetilde{\mathcal{Y}}$ , by  $A_{\mathcal{Y}_2}$  the strict transform on  $\mathcal{Y}_2$  of  $A_{\widetilde{\mathcal{Y}}}$ , and by  $A_{\mathcal{Y}_3}$  the image on  $\mathcal{Y}_3$  of  $A_{\mathcal{Y}_2}$ . It holds:*

- (1)  $A \rightarrow A_{\widetilde{\mathcal{Y}}}$  is the contraction of  $E_{AB} \simeq \mathbb{P}(T_{\mathbb{P}(V_3)})$  to  $\mathbb{P}(V_3)$ .  $\mathbb{P}(V_3)$  is the image of  $B$  by the morphism  $p_B$  as in Proposition 7.4.1 (2) and is equal to the singular locus of  $A_{\widetilde{\mathcal{Y}}}$ ,

- (2)  $A_{\mathcal{Y}_2} \rightarrow A_{\widetilde{\mathcal{Y}}}$  is the blow-up along the image on  $A_{\widetilde{\mathcal{Y}}}$  of the section  $s_A$  of  $A \rightarrow \mathbb{P}(V_3^*)$  associated to the injection  $\mathcal{O}_{\mathbb{P}(V_3^*)} \hookrightarrow \mathcal{O}_{\mathbb{P}(V_3^*)} \oplus T_{\mathbb{P}(V_3^*)}$ ,  
(3)  $A_{\mathcal{Y}_3} \simeq A_{\mathcal{Y}_2}$ . Moreover,  $\mathcal{P}_\rho|_{A_{\mathcal{Y}_3}}$  is isomorphic to  $\mathbb{P}(T_{\mathbb{P}(V_3)}(-1))$ , and  
(4) let  $\widehat{A} \rightarrow A$  be the blow-up of  $A$  along the section  $s_A$ . Then there exists a natural morphism  $\widehat{A} \rightarrow A_{\mathcal{Y}_2}$ , which is the blow-up of  $A_{\mathcal{Y}_2}$  along its singular locus.



*Proof.* Part (1) follows from Proposition 7.3.1.

Part (2) follows from the description of  $\rho$ -double lines as in Proposition 7.1.2 and Proposition 7.3.1 since  $\mathcal{Y}_2 \rightarrow \widetilde{\mathcal{Y}}$  is the blow-up along  $G_\rho$ .

Now we prove Part (3) and Part (4). Let  $\mathcal{I}_\rho$  be the ideal sheaf of the section  $s_A$  as in Part (2). Denote by  $H_A$  the pull back on  $A$  of  $\mathcal{O}_{\mathbb{P}(V_3^*)}(1)$ . Note that  $E_{AB}$  is the tautological divisor associated to  $\mathcal{O}_{\mathbb{P}(V_3^*)} \oplus T_{\mathbb{P}(V_3^*)}$ . Then, by a standard argument using the theorem of cohomology and base change, we see that the natural map  $H^0(A, \mathcal{O}_A(E_{AB} + 2H_A) \otimes \mathcal{I}_\rho) \otimes \mathcal{O}_A \rightarrow \mathcal{O}_A(E_{AB} + 2H_A) \otimes \mathcal{I}_\rho$  is surjective. Therefore the rational map defined by the linear system  $|\mathcal{O}_A(E_{AB} + 2H_A) \otimes \mathcal{I}_\rho|$  becomes a morphism on the blow-up  $\widehat{A}$  of  $A$  along the section. Since  $E_{AB} + 2H_A$  is the tautological divisor associated to  $\mathcal{O}_{\mathbb{P}(V_3^*)}(-2) \oplus T_{\mathbb{P}(V_3^*)}(-2) \simeq \mathcal{O}_{\mathbb{P}(V_3^*)}(-2) \oplus \Omega_{\mathbb{P}(V_3^*)}(1)$ , the target of the morphism is  $\mathbb{P}(H^0(T_{\mathbb{P}(V_3^*)}(-1))^*) \simeq \mathbb{P}(V_3)$ . Moreover, by this morphism, the divisor  $E_{AB} \simeq \mathbb{P}(T_{\mathbb{P}(V_3)})$  is contracted to  $\mathbb{P}(V_3)$ , hence the morphism  $\widehat{A} \rightarrow \mathbb{P}(V_3)$  is surjective and factors through  $A_{\mathcal{Y}_2}$ . Since  $\rho(A_{\mathcal{Y}_2}) = 2$ , the two morphisms  $A_{\mathcal{Y}_2} \rightarrow A_{\widetilde{\mathcal{Y}}}$  and  $A_{\mathcal{Y}_2} \rightarrow \mathbb{P}(V_3)$  are the only nontrivial morphisms from  $A_{\mathcal{Y}_2}$ . Therefore, since  $A_{\mathcal{Y}_2} \rightarrow A_{\mathcal{Y}_3}$  is birational and is different from  $A_{\mathcal{Y}_2} \rightarrow A_{\widetilde{\mathcal{Y}}}$ , it must be an isomorphism. Hence  $\mathcal{P}_\rho|_{A_{\mathcal{Y}_3}}$  is isomorphic to the exceptional divisor of  $A_{\mathcal{Y}_2} \rightarrow A_{\widetilde{\mathcal{Y}}}$ , and the latter is isomorphic to  $\mathbb{P}(T_{\mathbb{P}(V_3)}(-1))$ . Since  $\pi_{\mathcal{Y}_3}|_{\mathcal{P}_\rho \cap A_{\mathcal{Y}_3}}$  is surjective, so is  $\pi_{A_{\mathcal{Y}_3}}$ .  $\square$

For a locally free sheaf  $\mathcal{E}$  on  $\widetilde{\mathcal{Y}}$ , we denote by  $\mathcal{E}_A$  and  $\mathcal{E}_B$  its pull-backs on  $A$  and  $B$ , respectively unless stated otherwise. Denote by  $H_A$  the pull back on  $A$  of  $\mathcal{O}_{\mathbb{P}(V_3^*)}(1)$ , and  $F_A$  and  $F_B$  the pull-backs of  $F_{\widetilde{\mathcal{Y}}}$  to  $A$  and  $B$ , respectively.

**Lemma 7.5.2.** *It holds that*

- (1)  $F_A \simeq -(E_{AB} + 2H_A)$ ,  $(\widetilde{\mathcal{S}}_L)_A \simeq \widetilde{\mathcal{Q}}_A \simeq \mathcal{O}_A \oplus \mathcal{V}$ , and  $\widetilde{\mathcal{T}}_A \simeq \mathcal{O}_A^{\oplus 2} \oplus \mathcal{V}$ , where  $\mathcal{V}$  is a locally free sheaf obtained as a unique nonsplit extension

$$0 \rightarrow \mathcal{O}_A(H_A + E_{AB}) \rightarrow \mathcal{V} \rightarrow \mathcal{O}_A(H_A) \rightarrow 0.$$

- (2)  $\mathcal{O}_B(F_B) \simeq p_B^* \mathcal{O}_{\mathbb{P}(V_3)}(-1)$ ,  $(\widetilde{\mathcal{S}}_L)_B \simeq \widetilde{\mathcal{Q}}_B \simeq \mathcal{O}_B \oplus p_B^* T_{\mathbb{P}(V_3)}(-1)$ , and  $\widetilde{\mathcal{T}}_B \simeq \mathcal{O}_B^{\oplus 2} \oplus p_B^* T_{\mathbb{P}(V_3)}(-1)$ , where  $p_B: B \rightarrow \mathbb{P}(V_3)$  is as in Proposition 7.4.1 (2).

*Proof.*

**Step 1.**  $\det(\tilde{\mathcal{S}}_L)_A = \det \tilde{\mathcal{Q}}_A = E_{AB} + 2H_A$ .

By (6.4), we have only to determine  $\tilde{\mathcal{Q}}_A$ . By the proof of Lemma 7.5.1 (3) and (4), we have  $\hat{E}_{AB} + 2H_{\hat{A}} - G = L_{\hat{A}}$ , where  $H_{\hat{A}}$  and  $\hat{E}_{AB}$  are the pull-backs on  $\hat{A}$  of  $H_A$  and  $E_{AB}$ , respectively,  $L_{\hat{A}}$  is the pull-back of  $\mathcal{O}_{\mathbb{P}(V_3)}(1)$ , and  $G$  is the exceptional divisor for  $\hat{A} \rightarrow A$ . Therefore, by Proposition 6.4.5, we have  $\det \tilde{\mathcal{Q}}_A = E_{AB} + 2H_A$  since  $M_{\tilde{\mathcal{Y}}}$  is trivial on a fiber of  $F_{\tilde{\mathcal{Y}}} \rightarrow G_{\tilde{\mathcal{Y}}}$  and  $G$  is the pull-back of the exceptional divisor  $F_{\rho}$  of  $\mathcal{Y}_2 \rightarrow \tilde{\mathcal{Y}}$ .

**Step 2.**  $F_A = -(E_{AB} + 2H_A)$ .

By Proposition 6.4.5, we have  $L_{\tilde{\mathcal{Y}}} = \det \tilde{\mathcal{Q}} - M_{\tilde{\mathcal{Y}}}$ , where  $L_{\tilde{\mathcal{Y}}}$  is the image of  $L_{\tilde{\mathcal{Y}}_2}$  on  $\tilde{\mathcal{Y}}$ . Therefore, by (6.14), we have  $K_{\tilde{\mathcal{Y}}} = -6 \det \tilde{\mathcal{Q}} + 4L_{\tilde{\mathcal{Y}}} = -2 \det \tilde{\mathcal{Q}} - 4M_{\tilde{\mathcal{Y}}}$ . Further, by Proposition 6.9.2 (2), we have  $-2 \det \tilde{\mathcal{Q}} - 4M_{\tilde{\mathcal{Y}}} = -10M_{\tilde{\mathcal{Y}}} + 2F_{\tilde{\mathcal{Y}}}$ . Therefore, since the pull-back of  $M_{\tilde{\mathcal{Y}}}$  to  $A$  is trivial, we have  $F_A = -\det \tilde{\mathcal{Q}}_A$ . Consequently, we obtain  $F_A = -(E_{AB} + 2H_A)$  by the first paragraph.

**Step 3.**  $(\tilde{\mathcal{S}}_L)_A \simeq \tilde{\mathcal{Q}}_A \simeq \mathcal{O}_A \oplus \mathcal{V}$ .

We investigate the restriction of the universal exact sequence (6.3) on  $A_{\mathcal{Y}_3}$ . Let  $\mathcal{S}_{A_{\mathcal{Y}_3}}$  and  $\mathcal{Q}_{A_{\mathcal{Y}_3}}$  be the restrictions of  $\mathcal{S}$  and  $\mathcal{Q}$ , respectively. Then we obtain

$$(7.11) \quad 0 \rightarrow \mathcal{S}_{A_{\mathcal{Y}_3}}^* \rightarrow \pi_{A_{\mathcal{Y}_3}}^*(T(-1)^{\wedge 2}|_{\mathbb{P}(V_3)}) \rightarrow \mathcal{Q}_{A_{\mathcal{Y}_3}} \rightarrow 0.$$

Note that

$$(7.12) \quad \begin{aligned} \wedge^2(T(-1)|_{\mathbb{P}(V_3)}) &\simeq \wedge^2(T_{\mathbb{P}(V_3)}(-1) \oplus V/V_3 \otimes \mathcal{O}_{\mathbb{P}(V_3)}) \simeq \\ &\mathcal{O}_{\mathbb{P}(V_3)}(1) \oplus V/V_3 \otimes T_{\mathbb{P}(V_3)}(-1) \oplus \wedge^2(V/V_3) \otimes \mathcal{O}_{\mathbb{P}(V_3)}. \end{aligned}$$

From now on, we consider the situation in Proposition 7.1.2 and use the notation there. We choose the basis of  $\bar{V}_3 = V_4/V_1$  as in there such that  $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2\}$  is a basis of  $V_3/V_1$ . Take the point  $[\mathbb{P}_q^2] \in A_{\mathcal{Y}_3}$  associated to a double line  $q$  with  $V_2 \subset V_3$ . Note that  $\mathbb{P}_q^2$  is the projectivization of the fiber  $\mathcal{S}_q^*$  of  $\mathcal{S}^*$  at  $[\mathbb{P}_q^2] \in G(3, T(-1)^{\wedge 2})$ . Since the fiber of the pull-back  $L_{A_{\mathcal{Y}_3}}$  of  $\mathcal{O}_{\mathbb{P}(V_3)}(1)$  at  $[\mathbb{P}_q^2]$  is generated by  $[\bar{\mathbf{e}}_1 \wedge \bar{\mathbf{e}}_2]$ ,  $L_{A_{\mathcal{Y}_3}}$  is a direct summand of  $(\mathcal{S}^*)_{A_{\mathcal{Y}_3}}$ . We write  $(\mathcal{S}^*)_{A_{\mathcal{Y}_3}} = (\mathcal{S}'^*)_{A_{\mathcal{Y}_3}} \oplus L_{A_{\mathcal{Y}_3}}$  with a locally free sheaf  $(\mathcal{S}'^*)_{A_{\mathcal{Y}_3}}$  of rank two on  $A_{\mathcal{Y}_3}$ . Moreover, the fiber of the pull-back of  $\wedge^2(V/V_3) \otimes \mathcal{O}_{\mathbb{P}(V_3)}$  at  $[\mathbb{P}_q^2]$  is generated by  $[\bar{\mathbf{e}}_3 \wedge \bar{\mathbf{e}}_4]$ ,  $(\mathcal{S}^*)_{A_{\mathcal{Y}_3}}$  is contained in  $L_{A_{\mathcal{Y}_3}} \oplus V/V_3 \otimes \pi_{A_{\mathcal{Y}_3}}^* T_{\mathbb{P}(V_3)}(-1)$ . Therefore, we obtain the following exact sequence from (7.11):

$$0 \rightarrow (\mathcal{S}'^*)_{A_{\mathcal{Y}_3}} \rightarrow V/V_3 \otimes \pi_{A_{\mathcal{Y}_3}}^* T_{\mathbb{P}(V_3)}(-1) \rightarrow \mathcal{Q}'_{A_{\mathcal{Y}_3}} \rightarrow 0,$$

where  $\mathcal{Q}_{A_{\mathcal{Y}_3}} \simeq \mathcal{Q}'_{A_{\mathcal{Y}_3}} \oplus \mathcal{O}_{A_{\mathcal{Y}_3}}$ . The pull-back  $(\mathcal{S}'^*)_{\hat{A}}$  on  $\hat{A}$  of  $(\mathcal{S}'^*)_{A_{\mathcal{Y}_3}}$  contains a subbundle of rank one whose fiber at a point over  $[\mathbb{P}_q^2]$  is generated by  $\bar{\mathbf{e}}_1 \wedge \bar{\mathbf{e}}_3$ . Note that  $\bar{\mathbf{e}}_1 \wedge \bar{\mathbf{e}}_3$  is a basis of  $V_2/V_1 \otimes V_4/V_3$ . Since  $V_3$  and  $V_4$  are fixed, the vector space  $V_4/V_3$  is the fiber of the trivial bundle on  $\hat{A}$ . Since  $V_1$  is the fiber of  $-L_{\hat{A}}$  and  $V_2$  is the fiber of the pull-back of  $\Omega_{\mathbb{P}(V_3^*)}(1)$ , we see that  $V_2/V_1$  is the fiber of  $\mathcal{O}_{\hat{A}}(-H_{\hat{A}} + L_{\hat{A}})$  by taking the determinants. Therefore  $(\mathcal{S}'^*)_{\hat{A}}(-L_{\hat{A}})$  is presented as an extension as follows:

$$0 \rightarrow \mathcal{O}(-H_{\hat{A}}) \rightarrow (\mathcal{S}'^*)_{\hat{A}}(-L_{\hat{A}}) \rightarrow \mathcal{O}(-H_{\hat{A}} - \hat{E}_{AB}) \rightarrow 0,$$

where the quotient is determined by taking determinants. Since  $\mathcal{O}(-H_{\hat{A}})$ ,  $(\mathcal{S}'^*)_{\hat{A}}(-L_{\hat{A}})$ , and  $\mathcal{O}(-H_{\hat{A}} - \hat{E}_{AB})$  are the pull-backs of locally free sheaves on  $A$ , we have

$$0 \rightarrow \mathcal{O}_A(-H_A) \rightarrow (\tilde{\mathcal{S}}_L^*)_A \rightarrow \mathcal{O}_A(-H_A - E_{AB}) \rightarrow 0,$$

where  $(\mathcal{S}'^*)_{\hat{A}}(-L_{\hat{A}})$  is the pull-back of  $(\tilde{\mathcal{S}}_L^*)_A$ . The sequence does not split since  $(\tilde{\mathcal{S}}_L^*)_A$  is the pull-back of a locally free sheaf on  $A_{\mathcal{Y}}$  but  $H_A$  is not. Since

$$\begin{aligned} \text{Ext}^1(\mathcal{O}_A(-H_A - E_{AB}), \mathcal{O}_A(-H_A)) &\simeq H^1(A, \mathcal{O}_A(E_{AB})) \simeq \\ &H^1(\mathbb{P}(V_3^*), \mathcal{O}_{\mathbb{P}(V_3^*)} \oplus \Omega_{\mathbb{P}(V_3^*)}^1) \simeq \mathbb{C}, \end{aligned}$$

we see that  $(\tilde{\mathcal{S}}_L^*)_A \simeq \mathcal{V}$  as in the statement, and  $(\tilde{\mathcal{S}}_L)_A \simeq \mathcal{V} \oplus \mathcal{O}_A$ .

Let  $\mathcal{Q}'_{\hat{A}}$  be the pull-back on  $\hat{A}$  of  $\mathcal{Q}'_{A_{\mathcal{Y}_3}}$ . We see that there is a surjective map from  $\mathcal{Q}'_{\hat{A}}$  to the invertible sheaf whose fiber at a point over  $[\mathbb{P}_q^2]$  is generated by  $\bar{\mathbf{e}}_2 \wedge \bar{\mathbf{e}}_4$  by the description of  $\mathbb{P}_q^2$  as in Proposition 7.1.2. Since  $\bar{\mathbf{e}}_2 \wedge \bar{\mathbf{e}}_4$  is a basis of  $V_3/V_2 \otimes V/V_4$ , the invertible sheaf is  $\mathcal{O}_{\hat{A}}(H_{\hat{A}})$ . Therefore  $\mathcal{Q}'_{\hat{A}}$  is presented as an extension:

$$0 \rightarrow \mathcal{O}(H_{\hat{A}} + \hat{E}_{AB}) \rightarrow \mathcal{Q}'_{\hat{A}} \rightarrow \mathcal{O}(H_{\hat{A}}) \rightarrow 0,$$

where the kernel is determined by taking determinants. Therefore we see that  $\mathcal{Q}'_{\hat{A}}$  is also the pull-back of  $\mathcal{V}$  and  $\tilde{\mathcal{Q}}_A \simeq \mathcal{V} \oplus \mathcal{O}_A$  as we determine  $(\tilde{\mathcal{S}}_L)_A$ .

**Step 4.**  $\tilde{\mathcal{T}}_A \simeq \mathcal{O}_A^{\oplus 2} \oplus \mathcal{V}$ .

By Lemma 7.5.1 (3),  $\mathcal{P}_\rho|_{A_{\mathcal{Y}_3}} \simeq \mathbb{P}(T_{\mathbb{P}(V_3)}(-1))$  and this lifts to  $\mathcal{Y}_2$  isomorphically. Therefore, restricting (6.18) to  $A_{\mathcal{Y}_2}$ , we obtain

$$(7.13) \quad 0 \rightarrow \mathcal{T}_{A_{\mathcal{Y}_2}}^* \rightarrow \pi_{A_{\mathcal{Y}_2}}^*(\Omega_{\mathbb{P}(V_3)}^1(1) \oplus \mathcal{O}_{\mathbb{P}(V_3)}^{\oplus 2}) \rightarrow \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}(V_3)}(-1))}(1) \rightarrow 0,$$

where  $\mathcal{T}_{A_{\mathcal{Y}_2}}^*$  and  $\pi_{A_{\mathcal{Y}_2}}$  are the restrictions to  $A_{\mathcal{Y}_2}$  of  $\mathcal{T}^*$  and  $\pi_{\mathcal{Y}_2}$  respectively. Since  $h^0(\mathcal{O}_{\mathbb{P}(T_{\mathbb{P}(V_3)}(-1))}(1)) = h^0(\Omega_{\mathbb{P}(V_3)}^1(1)) = 0$ , we have

$$\mathcal{T}_{A_{\mathcal{Y}_2}}^* \simeq \mathcal{O}_{A_{\mathcal{Y}_2}}^{\oplus 2} \oplus \mathcal{V}',$$

where  $\mathcal{V}'$  is the kernel of the map

$$(7.14) \quad \pi_{A_{\mathcal{Y}_2}}^* \Omega_{\mathbb{P}(V_3)}^1(1) \rightarrow \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}(V_3)}(-1))}(1).$$

Again, we consider the situation in Proposition 7.1.2 and use the notation there. We choose the basis of  $\bar{V}_3 = V_4/V_1$  as in there such that  $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2\}$  is a basis of  $V_3/V_1$ . Take the point  $[\mathbb{P}_q^2] \in A_{\mathcal{Y}_3}$  associated to a double line  $q$  with  $V_2 \subset V_3$ . Note that the fiber of  $\pi_{A_{\mathcal{Y}_3}}^* \Omega_{\mathbb{P}(V_3)}^1(1)$  at  $[\mathbb{P}_q^2] \in A_{\mathcal{Y}_3}$  is isomorphic to  $(V_3/V_1)^*$ . Similarly to the argument below (7.12), the vector space  $(V_3/V_2)^*$  can be considered to be the fiber of  $\mathcal{O}_{\hat{A}}(-H_{\hat{A}})$  at a point over  $[V_2] \in \mathbb{P}(V_3)$ . Therefore we have a natural injection

$$\mathcal{O}_{\hat{A}}(-H_{\hat{A}}) \hookrightarrow \pi_{\hat{A}}^* \Omega_{\mathbb{P}(V_3)}^1(1),$$

where the cokernel  $\mathcal{K}_1$  is an invertible sheaf and  $\pi_{\hat{A}}$  is the naturally induced map  $\hat{A} \rightarrow \mathbb{P}(V_3)$ . By taking the determinants, we see that  $\mathcal{K}_1 = \mathcal{O}_{\hat{A}}(H_{\hat{A}} - L_{\hat{A}})$ . We show that the composite  $\mathcal{O}_{\hat{A}}(-H_{\hat{A}}) \rightarrow \mathcal{O}_{\mathbb{P}(T_{\mathbb{P}(V_3)}(-1))}(1)$  of this map with the pull-back of (7.14) is zero, where we note that  $\mathbb{P}(T_{\mathbb{P}(V_3)}(-1))$  lifts isomorphically on  $\hat{A}$ . Indeed,  $H_{\mathbb{P}(\Omega_{\mathbb{P}(V_3)}^1(1))}$  is nothing but the pull-back of  $H_A$  by the map  $\hat{A} \rightarrow A$ . Since  $H_{\mathbb{P}(T_{\mathbb{P}(V_3)}(-1))} = H_{\mathbb{P}(\Omega_{\mathbb{P}(V_3)}^1(1))} - L$ , where  $L$  is the pull-back of  $\mathcal{O}_{\mathbb{P}(V_3)}(1)$  to  $\mathbb{P}(T_{\mathbb{P}(V_3)}(-1))$ , we have only to show  $h^0(2H_{\mathbb{P}(\Omega_{\mathbb{P}(V_3)}^1(1))} - L) = 0$ , which follows

from the Bott theorem 2.0.1 since  $H^0(2H_{\mathbb{P}(\Omega_{\mathbb{P}(V_3)}^1(1))} - L) \simeq H^0(S^2(T_{\mathbb{P}(V_3)}(-1)) \otimes \mathcal{O}_{A_{\mathcal{Q}_2}}(-1))$ . Therefore we have an injection  $\mathcal{O}_{\hat{A}}(-H_{\hat{A}}) \hookrightarrow \mathcal{V}'_{\hat{A}}$ , where  $\mathcal{V}'_{\hat{A}}$  is the pull-back of  $\mathcal{V}'$  on  $\hat{A}$  and the cokernel  $\mathcal{K}_2$  has the following expression as an extension:

$$0 \rightarrow \mathcal{K}_2 \rightarrow \mathcal{O}_{\hat{A}}(H_{\hat{A}} - L_{\hat{A}}) \rightarrow H_{\mathbb{P}(T_{\mathbb{P}(V_3)}(-1))} \rightarrow 0.$$

Taking  $\mathcal{E}xt^\bullet(-, \mathcal{O}_{\hat{A}})$  of this exact sequence, we see that  $\mathcal{K}_2$  is also an invertible sheaf by [Ha, III, Ex 6.6]. By taking the determinants, we see that  $\mathcal{K}_2 = \mathcal{O}_{\hat{A}}(H_{\hat{A}} - L_{\hat{A}} - F_\rho|_{\hat{A}})$ , where  $F_\rho|_{\hat{A}}$  is the pull-back of  $F_\rho$ . Since  $M_{\mathcal{Q}_2}|_{A_{\mathcal{Q}_2}} = 0$ , we have  $\det \mathcal{Q}_{\hat{A}} = L_{\hat{A}} + F_\rho|_{\hat{A}}$  by Proposition 6.4.5, where  $\mathcal{Q}_{\hat{A}}$  is the pull-backs of  $\mathcal{Q}$ . Therefore,  $\mathcal{K}_2 = \mathcal{O}_{\hat{A}}(H_{\hat{A}} - \det \mathcal{Q}_{\hat{A}}) = \mathcal{O}_{\hat{A}}(-H_{\hat{A}} - \hat{E}_{AB})$ , where the second equality follows from the first paragraph. Therefore  $\mathcal{V}'_{\hat{A}}$  has the following expression as an extension:

$$0 \rightarrow \mathcal{O}_{\hat{A}}(-H_{\hat{A}}) \rightarrow \mathcal{V}'_{\hat{A}} \rightarrow \mathcal{O}_{\hat{A}}(-H_{\hat{A}} - \hat{E}_{AB}) \rightarrow 0.$$

Consequently, we have  $\tilde{\mathcal{T}}_A \simeq \mathcal{O}_A^{\oplus 2} \oplus \mathcal{V}$  as we determine  $(\tilde{\mathcal{S}}_L)_A$ .

**Step 5.**  $F_B$ ,  $(\tilde{\mathcal{S}}_L)_B$ ,  $\tilde{\mathcal{Q}}_B$  and  $\tilde{\mathcal{T}}_B$ .

By Lemma 7.5.1 (1), the image of  $B$  on  $F_{\tilde{\mathcal{Y}}}$  is the  $\mathbb{P}(V_3)$  contained in  $A_{\tilde{\mathcal{Y}}}$ . Therefore,  $F_B$ ,  $(\tilde{\mathcal{S}}_L)_B$ ,  $\tilde{\mathcal{Q}}_B$  and  $\tilde{\mathcal{T}}_B$  are the pull-back of the restrictions of  $F_{\tilde{\mathcal{Y}}}$ ,  $\tilde{\mathcal{Q}}$ , and  $\tilde{\mathcal{T}}$  to  $\mathbb{P}(V_3)$ . Since  $F_A|_{E_{AB}} \simeq -(E_{AB} + 2H_A)|_{E_{AB}}$  by Step 2, and this is the pull-back of  $\mathcal{O}_{\mathbb{P}(V_3)}(-1)$ , we have  $F_B = p_B^* \mathcal{O}_{\mathbb{P}(V_3)}(-1)$ . Since  $\tilde{\mathcal{T}}_A \simeq \mathcal{O}_A \oplus (\tilde{\mathcal{S}}_L)_A \simeq \mathcal{O}_A \oplus \tilde{\mathcal{Q}}_A$  as above, we have  $\tilde{\mathcal{T}}_B \simeq \mathcal{O}_B \oplus (\tilde{\mathcal{S}}_L)_B \simeq \mathcal{O}_B \oplus \tilde{\mathcal{Q}}_B$ . Thus we have only to determine  $\tilde{\mathcal{T}}_B$ . Recall that  $\mathbb{P}(V_3)$  in  $A_{\tilde{\mathcal{Y}}}$  parameterizes the planes spanned by  $\sigma$ -conics by Proposition 7.3.1. Therefore  $\mathbb{P}(V_3)$  is disjoint from the locus of  $\rho$ -conics, and then, by (6.19), we have  $\tilde{\mathcal{T}}_B \simeq p_B^*(T(-1)|_{\mathbb{P}(V_3)}) \simeq \mathcal{O}_B^{\oplus 2} \oplus p_B^*(T_{\mathbb{P}(V_3)}(-1))$ .  $\square$

## 8. THE DERIVED CATEGORY OF THE RESOLUTION $\widetilde{\mathcal{Y}}$ OF THE DOUBLE SYMMETROID

In the previous sections, we have obtained a desingularization  $\rho_{\widetilde{\mathcal{Y}}} : \widetilde{\mathcal{Y}} \rightarrow \mathcal{Y}$ . To have  $\widetilde{\mathcal{Y}}$  we have started with the Hilbert scheme of conics in  $G(3, V)$  and arrived at the relative Grassmann bundle  $\mathcal{Y}_3 = G(3, T(-1)^{\wedge 2})$  over  $\mathbb{P}(V)$ . The disjoint union of the orthogonal Grassmann varieties  $\mathcal{P}_\sigma$  and  $\mathcal{P}_\rho$  is naturally defined as the maximally isotropic subspaces with respect to the quadric  $G(2, T(-1)) \subset \mathbb{P}(T(-1)^{\wedge 2})$ . Here we reproduce the relevant part of the diagram (6.1) for our calculations in this section:

$$\begin{array}{ccccc} \mathcal{P}_\rho & \xleftarrow{\quad} & F_\rho & \xrightarrow{\quad} & G_\rho \\ \cap & & \cap & & \cap \\ \mathcal{Y}_3 & \xleftarrow{\quad \rho_{\mathcal{Y}_2} \quad} & \mathcal{Y}_2 & \xrightarrow{\quad \tilde{\rho}_{\mathcal{Y}_2} \quad} & \widetilde{\mathcal{Y}} \end{array}$$

where  $\rho_{\mathcal{Y}_2}$  is the blow-up along  $\mathcal{P}_\rho$  with its exceptional divisor  $F_\rho$ , and  $\tilde{\rho}_{\mathcal{Y}_2}$  is the contraction of  $F_\rho$  to another direction. In Subsections 6.4–6.6, we have seen that this process is nothing but a step in the two ray game.

Recall the universal exact sequence of  $\mathcal{Y}_3$ ,

$$0 \rightarrow \mathcal{S}^* \rightarrow \pi_{\mathcal{Y}_3}^*(T(-1)^{\wedge 2}) \rightarrow \mathcal{Q} \rightarrow 0,$$

and also our definition of the sheaf  $\mathcal{T}^* = \text{Ker} \{ \pi_{\mathcal{Y}_2}^* \Omega(1) \rightarrow (\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(1) \}$  on  $\mathcal{Y}_2$  in (6.18), where  $\pi_{\mathcal{Y}_2}$  represents the composite morphism  $\mathcal{Y}_2 \rightarrow \mathcal{Y}_3 \rightarrow \mathbb{P}(V)$ . Then, we have defined the sheaves  $\tilde{\mathcal{S}}_L$ ,  $\tilde{\mathcal{Q}}$  and  $\tilde{\mathcal{T}}$  on  $\widetilde{\mathcal{Y}}$  by the properties

$$(\rho_{\mathcal{Y}_2}^* \mathcal{S})(L_{\mathcal{Y}_2}) = \tilde{\rho}_{\mathcal{Y}_2}^* \tilde{\mathcal{S}}_L, \quad \rho_{\mathcal{Y}_2}^* \mathcal{Q} = \tilde{\rho}_{\mathcal{Y}_2}^* \tilde{\mathcal{Q}}, \quad \mathcal{T} = \tilde{\rho}_{\mathcal{Y}_2}^* \tilde{\mathcal{T}},$$

where  $L_{\mathcal{Y}_2} = \pi_{\mathcal{Y}_2}^* \mathcal{O}(1)$ . We also use  $M_{\widetilde{\mathcal{Y}}}$  for the pull-back of  $\mathcal{O}_{\mathcal{H}}(1)$  by the composite morphism  $\widetilde{\mathcal{Y}} \rightarrow \mathcal{Y} \rightarrow \mathcal{H}$ . These sheaves on  $\widetilde{\mathcal{Y}}$  are studied in this section.

### 8.1. Constructing a Lefschetz collection in $\mathcal{D}^b(\widetilde{\mathcal{Y}})$ .

We will show that the sheaves  $\tilde{\mathcal{S}}_L$ ,  $\tilde{\mathcal{Q}}$ ,  $\tilde{\mathcal{T}}$  and  $\mathcal{O}_{\widetilde{\mathcal{Y}}}$  define a Lefschetz collection in  $\mathcal{D}^b(\widetilde{\mathcal{Y}})$ , which we expect to be the ‘dual’ to the dual Lefschetz collection in  $\mathcal{D}^b(\widetilde{\mathcal{X}})$  obtained in Theorem 3.4.4.

**Theorem 8.1.1.** (1) *Let*

$$(\mathcal{E}_3, \mathcal{E}_2, \mathcal{E}_{1a}, \mathcal{E}_{1b}) = (\tilde{\mathcal{S}}_L^*, \tilde{\mathcal{T}}^*, \mathcal{O}_{\widetilde{\mathcal{Y}}}, \tilde{\mathcal{Q}}^*(M_{\widetilde{\mathcal{Y}}}))$$

*be an ordered collection of sheaves on  $\widetilde{\mathcal{Y}}$ . Then  $(\tilde{\mathcal{B}}_i)_{1 \leq i \leq 4} := (\mathcal{E}_3, \mathcal{E}_2, \mathcal{E}_{1a}, \mathcal{E}_{1b})$  is a strongly exceptional collection, namely, it satisfies*

$$H^\bullet(\tilde{\mathcal{B}}_i^* \otimes \tilde{\mathcal{B}}_j) = 0 \text{ for } 1 \leq i, j \leq 4 \text{ and } \bullet > 0$$

$$\text{and } H^0(\tilde{\mathcal{B}}_i^* \otimes \tilde{\mathcal{B}}_j) = 0 \text{ (} i > j \text{), } H^0(\tilde{\mathcal{B}}_i^* \otimes \tilde{\mathcal{B}}_i) = \mathbb{C} \text{ (} 1 \leq i \leq 4 \text{).}$$

(2) *For  $i < j$ ,  $H^0(\tilde{\mathcal{B}}_i^* \otimes \tilde{\mathcal{B}}_j)$  are given by*

$$\begin{aligned} H^0(\mathcal{E}_3^* \otimes \mathcal{E}_2) &\simeq V, \quad H^0(\mathcal{E}_3^* \otimes \mathcal{E}_{1a}) \simeq \wedge^2 V, \quad H^0(\mathcal{E}_3^* \otimes \mathcal{E}_{1b}) \simeq S^2 V, \\ H^0(\mathcal{E}_2^* \otimes \mathcal{E}_{1a}) &\simeq V, \quad H^0(\mathcal{E}_2^* \otimes \mathcal{E}_{1b}) \simeq V, \quad H^0(\mathcal{E}_{1a}^* \otimes \mathcal{E}_{1b}) \simeq 0, \end{aligned}$$

and may be summarized in the following diagram:

$$(8.1) \quad \begin{array}{ccc} & \xrightarrow{\wedge^2 V} & \circ \mathcal{E}_{1a} \\ & \nearrow V & \nearrow V \\ \circ \mathcal{E}_3 & \xrightarrow{V} & \circ \mathcal{E}_2 \\ & \searrow V & \searrow V \\ & \xrightarrow{S^2 V} & \circ \mathcal{E}_{1b} \end{array}$$

(3) Set

$$\mathcal{D}_{\widetilde{\mathcal{Y}}} := \langle \mathcal{E}_3, \mathcal{E}_2, \mathcal{E}_{1a}, \mathcal{E}_{1b} \rangle \subset \mathcal{D}^b(\widetilde{\mathcal{Y}}).$$

Then

$$\mathcal{D}_{\widetilde{\mathcal{Y}}}, \mathcal{D}_{\widetilde{\mathcal{Y}}}(1), \dots, \mathcal{D}_{\widetilde{\mathcal{Y}}}(9)$$

is a Lefschetz collection, namely, for  $1 \leq i, j \leq 4$  and  $\bullet > 0$  it holds that

$$H^\bullet(\widetilde{\mathcal{B}}_i^* \otimes \widetilde{\mathcal{B}}_j(-t)) = 0 \quad (1 \leq t \leq 9),$$

where  $(-t)$  represents the twist by the sheaf  $\mathcal{O}_{\widetilde{\mathcal{Y}}}(-tM_{\widetilde{\mathcal{Y}}})$ .

The rest of this section is devoted to our proof of Theorem 8.1.1, where we compute the cohomology groups  $H^\bullet(\widetilde{\mathcal{B}}_i^* \otimes \widetilde{\mathcal{B}}_j(-t))$  ( $0 \leq t \leq 9$ ). Our strategy is to reduce the computations of cohomology groups on  $\widetilde{\mathcal{Y}}$  to those on  $\mathcal{Y}_3$  and use the Bott Theorem 2.0.1 for the  $G(3, 6)$ -bundle  $\mathcal{Y}_3 \rightarrow \mathbb{P}(V)$ . This idea works for small values of  $t$  as we formulate in Proposition 8.1.6 below.

**Lemma 8.1.2.**  $K_{\widetilde{\mathcal{Y}}} = -10M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}}$

*Proof.* We have  $K_{\mathcal{Y}} = -10M_{\mathcal{Y}}$  by Proposition 4.2.4. Then from Proposition 6.9.2 (2),  $K_{\widetilde{\mathcal{Y}}} = \rho_{\widetilde{\mathcal{Y}}}^* K_{\mathcal{Y}} + 2F_{\widetilde{\mathcal{Y}}} = -10M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}}$ .  $\square$

Let us introduce

$$\widetilde{\mathcal{C}}_{ij} = \widetilde{\mathcal{B}}_i^* \otimes \widetilde{\mathcal{B}}_j \quad (1 \leq i, j \leq 4)$$

for the ordered collection  $(\widetilde{\mathcal{B}}_i)_{1 \leq i \leq 4}$ .

**Lemma 8.1.3.** For  $\widetilde{\mathcal{C}} = \widetilde{\mathcal{C}}_{ij}$  as above, it holds

$$H^\bullet(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{C}}(-t)) \simeq H^{13-\bullet}(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{C}}^*(t-10))$$

for any integer  $t$ .

*Proof.* Note that the cohomology groups  $H^\bullet(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{C}}(-t))$  is Serre dual to

$$H^{13-\bullet}(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{C}}^*((t-10)M_{\widetilde{\mathcal{Y}}} + 2F_{\widetilde{\mathcal{Y}}}).$$

Considering the exact sequence

$$0 \rightarrow \widetilde{\mathcal{C}}^*((t-10)M_{\widetilde{\mathcal{Y}}} + (i-1)F_{\widetilde{\mathcal{Y}}}) \rightarrow \widetilde{\mathcal{C}}^*((t-10)M_{\widetilde{\mathcal{Y}}} + iF_{\widetilde{\mathcal{Y}}}) \rightarrow \widetilde{\mathcal{C}}^*((t-10)M_{\widetilde{\mathcal{Y}}} + iF_{\widetilde{\mathcal{Y}}})|_{F_{\widetilde{\mathcal{Y}}}} \rightarrow 0,$$

it suffices to show

$$(8.2) \quad H^{13-\bullet}(F_{\widetilde{\mathcal{Y}}}, \mathcal{A}_i) = 0 \text{ for } i = 1, 2 \text{ and any } \bullet,$$

where we set

$$\mathcal{A}_i := \tilde{\mathcal{C}}^*((t-10)M_{\tilde{\mathcal{Y}}} + iF_{\tilde{\mathcal{Y}}})|_{F_{\tilde{\mathcal{Y}}}}.$$

Recall the diagrams (7.8) and (7.10). Since the morphism  $\hat{F} \rightarrow F_{\tilde{\mathcal{Y}}}$  is finite, and  $\hat{F}$  has only rational singularities by its construction, we have only to show the vanishings of cohomology groups of the pull-back of  $\mathcal{A}_i$  on  $F^{(3)}$ . Since the morphism  $F^{(3)} \rightarrow \hat{G}'$  is flat by Proposition 7.4.1, we have only to show the vanishing of cohomology groups of the pull-back of  $\mathcal{A}_i$  to its fibers. By the upper semi-continuity of cohomology groups on fibers, it suffices to prove the vanishing on fibers over points of the diagonal subset of  $\hat{G}'$ . By Proposition 7.4.1, such fibers are of the form  $A \cup B$ . Note that the pull-back of  $M_{\tilde{\mathcal{Y}}}$  to the fibers is trivial. Therefore, by Lemma 7.5.2, we have only to show the vanishing of the following cohomology groups:

$$H^\bullet(A \cup B, \mathcal{C}_{A \cup B}^*(iF_{A \cup B})) \quad (i = 1, 2),$$

where  $\mathcal{C}_{A \cup B}$  and  $F_{A \cup B}$  are the pull-backs of  $\tilde{\mathcal{C}}$  and  $F_{\tilde{\mathcal{Y}}}$  to  $A \cup B$ , respectively. Tensoring  $\mathcal{C}_{A \cup B}^*(iF_{A \cup B})$  with the Mayor-Vietris sequence, we have

$$(8.3) \quad 0 \rightarrow \mathcal{C}_{A \cup B}^*(iF_{A \cup B}) \rightarrow \mathcal{C}_A^*(iF_A) \oplus \mathcal{C}_B^*(iF_B) \rightarrow \mathcal{C}_{A \cap B}^*(iF_{A \cap B}) \rightarrow 0,$$

where  $\mathcal{C}_A, \mathcal{C}_B$ , and  $\mathcal{C}_{A \cap B}$  are the restrictions of  $\mathcal{C}_{A \cup B}$  to  $A, B$  and  $A \cap B$  respectively, and  $F_{A \cap B}$  is the restriction of  $F_{A \cup B}$  to  $A \cap B$ . By Lemma 7.5.2 (1), we can easily show the vanishing of  $H^\bullet(A, \mathcal{C}_A^*(iF_A))$ . Moreover, by Lemma 7.5.2 (2), we easily see that the restriction maps  $H^\bullet(B, \mathcal{C}_B^*(iF_B)) \rightarrow H^\bullet(A \cap B, \mathcal{C}_{A \cap B}^*(iF_{A \cap B}))$  are isomorphisms. Therefore we have the vanishing of  $H^\bullet(A \cup B, \mathcal{C}_{A \cup B}^*(iF_{A \cup B}))$ .  $\square$

*Remark.* The above lemma supports the conjecture that  $\mathcal{D}_{\tilde{\mathcal{Y}}}, \mathcal{D}_{\tilde{\mathcal{Y}}}(1), \dots, \mathcal{D}_{\tilde{\mathcal{Y}}}(9)$  generates a strongly crepant categorical resolution (cf. [Ku3]).  $\square$

Let us introduce a sequence of sheaves on  $\mathcal{Y}_2$

$$(\mathcal{B}_i)_{1 \leq i \leq 4} = (\rho_{\mathcal{Y}_2}^* \mathcal{S}^*(-L_{\mathcal{Y}_2}), \mathcal{T}^*, \mathcal{O}_{\mathcal{Y}_2}, \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*),$$

and define  $\mathcal{C}_{ij} = \mathcal{B}_i^* \otimes \mathcal{B}_j$  for  $1 \leq i, j \leq 4$ . Note that  $\mathcal{B}_i$  have its corresponding form to  $\tilde{\mathcal{B}}_i$  for  $i = 1, 2, 3$ , but  $\mathcal{B}_4$  is defined by removing the twist of  $M_{\tilde{\mathcal{Y}}}$  from  $\tilde{\mathcal{B}}_4$ .

Let  $\tilde{\mathcal{D}}$  be a locally free sheaf on  $\tilde{\mathcal{Y}}$  and  $\mathcal{D} := \tilde{\rho}_{\mathcal{Y}_2}^* \tilde{\mathcal{D}}$ . Since  $\tilde{\rho}_{\mathcal{Y}_2}: \mathcal{Y}_2 \rightarrow \tilde{\mathcal{Y}}$  is a blow-up of a smooth variety, it holds that

$$(8.4) \quad H^\bullet(\tilde{\mathcal{Y}}, \tilde{\mathcal{D}}(-t)) \simeq H^\bullet(\mathcal{Y}_2, \mathcal{D}(-t)),$$

where  $(-t)$  on the right hand side represents the twist by  $\mathcal{O}_{\mathcal{Y}_2}(-tM_{\tilde{\mathcal{Y}}_2})$ . By Definitions 6.7.1, 6.7.3 and Lemma 8.1.3, for our cohomology calculations, it suffices to know

$$(8.5) \quad \begin{aligned} H^\bullet(\tilde{\mathcal{Y}}, \tilde{\mathcal{C}}_{i4}(-t)) &= H^\bullet(\mathcal{Y}_2, \mathcal{C}_{i4}(-t+1)) \quad (t = 0, 1, \dots, 6) \\ H^\bullet(\tilde{\mathcal{Y}}, \tilde{\mathcal{C}}_{4j}(-t)) &= H^\bullet(\mathcal{Y}_2, \mathcal{C}_{4j}(-t-1)) \quad (t = 0, 1, \dots, 4) \end{aligned}$$

for  $1 \leq i, j \leq 3$  and

$$(8.6) \quad H^\bullet(\tilde{\mathcal{Y}}, \tilde{\mathcal{C}}_{ij}(-t)) = H^\bullet(\mathcal{Y}_2, \mathcal{C}_{ij}(-t)) \quad (t = 0, 1, \dots, 5)$$

for  $1 \leq i, j \leq 3$  or  $i = j = 4$ .



**Lemma 8.1.4.** *For the computations of (8.5) and (8.6), we may replace  $\mathcal{T}$  by  $\pi_{\mathcal{Y}_2}^* T(-1)$  except in one case  $\mathcal{C}_{24}(1) = \mathcal{T} \otimes \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*(M_{\mathcal{Y}_2})$ . From now on we may use the following relations:*

$$\begin{aligned} \mathcal{C}_{42}(-t-1) &= (\rho_{\mathcal{Y}_2}^* \mathcal{Q} \otimes \pi_{\mathcal{Y}_2}^* \Omega(1))(-t-1), \\ \mathcal{C}_{24}(-t+1) &= (\pi_{\mathcal{Y}_2}^* T(-1) \otimes \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*)(-t+1) \quad (t \neq 0), \\ \mathcal{C}_{i2}(-t) &= (\mathcal{B}_i^* \otimes \pi_{\mathcal{Y}_2}^* \Omega(1))(-t) \quad (i = 1, 3), \\ \mathcal{C}_{2j}(-t) &= (\pi_{\mathcal{Y}_2}^* T(-1) \otimes \mathcal{B}_j)(-t) \quad (j = 1, 3), \\ \mathcal{C}_{22}(-t) &= (\pi_{\mathcal{Y}_2}^* T(-1) \otimes \pi_{\mathcal{Y}_2}^* \Omega(1))(-t). \end{aligned}$$

*Proof.* First we consider  $\mathcal{C}_{i2}(-t)$ . By the exact sequence (6.18), we have

$$0 \rightarrow \mathcal{T}^* \otimes \mathcal{D}(-t) \rightarrow \pi_{\mathcal{Y}_2}^* \Omega(1) \otimes \mathcal{D}(-t) \rightarrow (\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(1) \otimes \mathcal{D}(-t)|_{F_\rho} \rightarrow 0$$

for a locally free sheaf  $\mathcal{D}$  on  $\mathcal{Y}_2$ . For  $\mathcal{D} = \mathcal{O}_{\mathcal{Y}_2}$ ,  $\rho_{\mathcal{Y}_2}^* \mathcal{Q}$ ,  $\rho_{\mathcal{Y}_2}^* \mathcal{S}(L_{\mathcal{Y}_2})$  or  $\mathcal{T}$ , it holds that

$$H^\bullet(F_\rho, (\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(1) \otimes \mathcal{D}(-t)|_{F_\rho}) = 0 \text{ for any } t$$

by the Leray spectral sequence for  $\tilde{\rho}_{\mathcal{Y}_2}|_{F_\rho} : F_\rho \rightarrow G_\rho$  since  $\tilde{\rho}_{\mathcal{Y}_2}|_{F_\rho}$  is a  $\mathbb{P}^1$ -bundle and the restriction of  $(\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(1) \otimes \mathcal{D}(-t)|_{F_\rho}$  to its fiber is a direct sum of  $\mathcal{O}_{\mathbb{P}^1}(-1)$  by Lemma 6.5.1. Therefore we have

$$H^\bullet(\mathcal{Y}_2, \mathcal{T}^* \otimes \mathcal{D}(-t)) \simeq H^\bullet(\mathcal{Y}_2, \pi_{\mathcal{Y}_2}^* \Omega(1) \otimes \mathcal{D}(-t))$$

for  $\mathcal{D} = \mathcal{O}_{\mathcal{Y}_2}$ ,  $\rho_{\mathcal{Y}_2}^* \mathcal{Q}$ ,  $\rho_{\mathcal{Y}_2}^* \mathcal{S}(L_{\mathcal{Y}_2})$  or  $\mathcal{T}$  and for any  $t$ .

Next we consider  $\mathcal{C}_{2j}(-t)$  except  $\mathcal{C}_{24}(1)$ . By the exact sequence (6.19), we have

$$0 \rightarrow \pi_{\mathcal{Y}_2}^* T(-1) \otimes \mathcal{D}(-t) \rightarrow \mathcal{T} \otimes \mathcal{D}(-t) \rightarrow (\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(-1) \otimes \mathcal{D}(-tM_{\mathcal{Y}_2} + F_\rho)|_{F_\rho} \rightarrow 0$$

for a locally free sheaf  $\mathcal{D}$  on  $\mathcal{Y}_2$ . Set  $\mathcal{D} = \mathcal{O}_{\mathcal{Y}_2}$ ,  $\rho_{\mathcal{Y}_2}^* \mathcal{Q}^*$ ,  $\rho_{\mathcal{Y}_2}^* \mathcal{S}^*(-L_{\mathcal{Y}_2})$ , or  $\pi_{\mathcal{Y}_2}^* \Omega(1)$ . We show the vanishing of

$$(8.7) \quad H^\bullet(F_\rho, (\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(-1) \otimes \mathcal{D}(-tM_{\mathcal{Y}_2} + F_\rho)|_{F_\rho}).$$

By Proposition 6.3.2, we have only to treat the case where  $\mathcal{D} = \mathcal{O}_{\mathcal{Y}_2}$ ,  $\rho_{\mathcal{Y}_2}^* \mathcal{Q}^*$  or  $\pi_{\mathcal{Y}_2}^* \Omega(1)$ . In the case where  $0 \leq t \leq 4$ , the vanishing of (8.7) follows from the Leray spectral sequence for  $\rho_{\mathcal{Y}_2}|_{F_\rho} : F_\rho \rightarrow \mathcal{P}_\rho$  since  $\rho_{\mathcal{Y}_2}|_{F_\rho}$  is a  $\mathbb{P}^5$ -bundle and the restriction of  $(\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(-1) \otimes \mathcal{D}(-tM_{\mathcal{Y}_2} + F_\rho)|_{F_\rho}$  to its fiber is a direct sum of  $\mathcal{O}_{\mathbb{P}^5}(-(t+1))$  by Proposition 6.4.5. Therefore we may assume that  $t = 5$  from now on. The cohomology groups (8.7) is Serre dual to

$$(8.8) \quad H^{12-\bullet}(F_\rho, (\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(1) \otimes \mathcal{D}^*(\rho_{\mathcal{Y}_2}^*(-\det \mathcal{Q} - L_{\mathcal{Y}_3}))|_{F_\rho})$$

by (6.14) and Proposition 6.4.5. Since  $\rho_{\mathcal{Y}_2}$  is the blow-up of a smooth variety and  $\mathcal{D}$  is the pull-back of a locally free sheaf  $\overline{\mathcal{D}}$  on  $\mathcal{Y}_3$ , the cohomology groups (8.8) is isomorphic to

$$(8.9) \quad H^{12-\bullet}(\mathcal{P}_\rho, \mathcal{O}_{\mathbb{P}(T(-1))}(1) \otimes \overline{\mathcal{D}}^*(-\det \mathcal{Q} - L_{\mathcal{Y}_3})|_{\mathcal{P}_\rho}).$$

Using  $\mathcal{R}_\rho$  as in (6.9), we can write

$$\mathcal{O}_{\mathbb{P}(T(-1))}(1) \otimes \overline{\mathcal{D}}^*(-\det \mathcal{Q} - L_{\mathcal{Y}_3})|_{\mathcal{P}_\rho} = \begin{cases} \mathcal{O}_{\mathcal{P}_\rho}(-\det \mathcal{R}_\rho - 2L_{\mathcal{P}_\rho}) : \overline{\mathcal{D}} = \mathcal{O}_{\mathcal{Y}_3} \\ \mathcal{R}_\rho^*(-2L_{\mathcal{P}_\rho}) : \overline{\mathcal{D}} = \mathcal{Q}^* \\ \pi_{\mathcal{Y}_2}^* \Omega(1)(-\det \mathcal{R}_\rho - 2L_{\mathcal{P}_\rho}) : \overline{\mathcal{D}} = \pi_{\mathcal{Y}_3}^* \Omega(1) \end{cases}$$

(see Proposition 6.3.3 (1) and (2)). All the cohomology groups of the restriction of these sheaves to a fiber of  $\pi_{\mathcal{Y}_3}|_{\mathcal{P}_\rho} : \mathcal{P}_\rho \rightarrow \mathbb{P}(V)$  vanish, thus so does (8.9) by the Leray spectral sequence for  $\pi_{\mathcal{Y}_3}|_{\mathcal{P}_\rho}$ .  $\square$

By the following simple lemma, we can reduce most of the computations of cohomology groups to those on  $\mathcal{Y}_3$ :

**Lemma 8.1.5.** (1)  $R^q \rho_{\mathcal{Y}_2*} \mathcal{O}(tF_\rho) = 0$  for any  $t \leq 5$  and  $q > 0$ .  
 (2)  $\rho_{\mathcal{Y}_2*} \mathcal{O}(tF_\rho) = \mathcal{O}_{\mathcal{Y}_3}$  for  $t \geq 0$ .

*Proof.* Part (1) follows from the relative Kodaira vanishing theorem since  $tF_\rho - K_{\mathcal{Y}_2} \equiv_{\mathcal{Y}_3} (t-5)F_\rho$  is  $\rho_{\mathcal{Y}_2}$ -nef and  $\rho_{\mathcal{Y}_2}$ -big if  $t \leq 5$ .

Part (2) is well-known.  $\square$

Now define an ordered collection of sheaves on  $\mathcal{Y}_3$ ,

$$(\overline{\mathcal{B}}_i)_{1 \leq i \leq 4} = (\mathcal{S}(L_{\mathcal{Y}_3})^*, \pi_{\mathcal{Y}_3}^* \Omega(1), \mathcal{O}_{\mathcal{Y}_3}, \mathcal{Q}^*),$$

and set  $\overline{\mathcal{C}}_{ij} = \overline{\mathcal{B}}_i^* \otimes \overline{\mathcal{B}}_j$ .

**Proposition 8.1.6.** *The cohomology groups on  $\mathcal{Y}_2$  in the r.h.s. of (8.5) and (8.6) can be evaluated by using*

$$(8.10) \quad H^\bullet(\mathcal{Y}_2, \mathcal{C}_{ij}(-t)) \simeq H^\bullet(\mathcal{Y}_3, \overline{\mathcal{C}}_{ij}(-t \det \mathcal{Q} + tL_{\mathcal{Y}_3})) \quad (t = 0, 1, \dots, 5)$$

for  $1 \leq i, j \leq 4$  except the cases of  $\mathcal{C}_{i4}(1)$  ( $1 \leq i \leq 3$ ). (We need to read  $t-1$  and  $t+1$  in the r.h.s. of (8.5) as  $t$ .)

*Proof.* Note that we may assume that the relation  $\mathcal{C}_{ij} = \rho_{\mathcal{Y}_2}^* \overline{\mathcal{C}}_{ij}$  holds for  $\mathcal{C}_{ij}(-t)$  except  $\mathcal{C}_{24}(1)$  by Lemma 8.1.4. Then by the Leray spectral sequence for  $\rho_{\mathcal{Y}_2}$ , and Proposition 6.4.5 and Lemma 8.1.5, we have the claimed isomorphisms for the range of  $t$  in (8.5) and (8.6) except the cases  $\mathcal{C}_{i4}(1)$  ( $1 \leq i \leq 3$ ), for which  $t = -1$ .  $\square$

## 8.2. Calculating cohomologies of coherent sheaves on $\mathcal{Y}_3$ .

Here first we calculate the r.h.s. of (8.10) postponing the cases  $\mathcal{C}_{i4}(1)$  ( $1 \leq i \leq 3$ ) to the latter half of this subsection. We use the Leray spectral sequence associated with  $\pi_{\mathcal{Y}_3}: \mathcal{Y}_3 \rightarrow \mathbb{P}(V)$ .

Let  $G \simeq G(3, 6)$  be a fiber of  $\pi_{\mathcal{Y}_3}$ . Since  $L_{\mathcal{Y}_3}$ ,  $\pi_{\mathcal{Y}_3}^* T(-1)$  and  $\pi_{\mathcal{Y}_3}^* \Omega(1)$  is the pull-back of the sheaves on  $\mathbb{P}(V)$ , and  $\det Q|_G = \mathcal{O}_G(1)$ , the restriction of  $\overline{\mathcal{C}}_{ij}(-t \det Q + tL_{\mathcal{Y}_3})$  to  $G$  is given by the direct sum of the following sheaves:

$$(8.11) \quad \begin{aligned} & \mathcal{S}|_G \otimes \mathcal{S}^*|_G(-t), \quad \mathcal{S}|_G(-t), \quad \mathcal{S}|_G \otimes \mathcal{Q}^*|_G(-t), & (0 \leq t \leq 5) \\ & \mathcal{S}^*|_G(-t), \quad \mathcal{O}_G(-t), \quad \mathcal{Q}^*|_G(-t), \quad \mathcal{Q}|_G \otimes \mathcal{Q}^*|_G(-t), & (0 \leq t \leq 5) \\ & \mathcal{Q}|_G \otimes \mathcal{S}^*|_G(-t), \quad \mathcal{Q}|_G(-t) & (1 \leq t \leq 5) \end{aligned}$$

where  $(-t)$  represents the twist by  $\mathcal{O}_G(-t)$ .

**Lemma 8.2.1.** *All the cohomology groups of the sheaves in (8.11) vanish except*

$$H^0(\mathcal{S}|_G \otimes \mathcal{S}^*|_G), \quad H^0(\mathcal{S}|_G), \quad H^0(\mathcal{O}_G), \quad \text{and} \quad H^0(\mathcal{Q}|_G \otimes \mathcal{Q}^*|_G).$$

*Proof.* These claims follow by the Bott theorem 2.0.1.  $\square$

**Proposition 8.2.2.** *Consider the cohomologies  $H^\bullet(\mathcal{Y}_2, \mathcal{C}_{ij}(-t))$  except the cases  $\mathcal{C}_{i4}(1)$  ( $1 \leq i \leq 3$ ) as in Proposition 8.1.6. The r.h.s. of (8.10) vanishes except possibly*

$$(8.12) \quad H^\bullet(\mathcal{Y}_2, \mathcal{C}_{ij}) \simeq H^\bullet(\mathbb{P}(V), \pi_{\mathcal{Y}_3*} \overline{\mathcal{C}}_{ij}) \quad \text{with } 1 \leq i, j \leq 3 \text{ or } i = j = 4.$$

*Proof.* This follows from the isomorphisms (8.10) and Lemma 8.2.1.  $\square$

For the evaluations of the r.h.s. of (8.12), we use the Bott theorem for  $\pi_{\mathcal{Y}_3}: \mathcal{Y}_3 \rightarrow \mathbb{P}(V)$ . All non-vanishing sheaves turns out to be

$$\begin{array}{ccc} \pi_{\mathcal{Y}_3*}\overline{\mathcal{C}}_{11}, & \pi_{\mathcal{Y}_3*}\overline{\mathcal{C}}_{12}, & \pi_{\mathcal{Y}_3*}\overline{\mathcal{C}}_{13} \\ \pi_{\mathcal{Y}_3*}\overline{\mathcal{C}}_{22}, & \pi_{\mathcal{Y}_3*}\overline{\mathcal{C}}_{23} & \pi_{\mathcal{Y}_3*}\overline{\mathcal{C}}_{33} \end{array} \simeq \begin{array}{ccc} \mathcal{O}, & T(-1)^{\wedge 2} \otimes \Omega(1), & T(-1)^{\wedge 2} \\ T(-1) \otimes \Omega(1), & T(-1) & \mathcal{O} \end{array}$$

and  $\pi_{\mathcal{Y}_3*}\overline{\mathcal{C}}_{44} \simeq \mathcal{O}$ .

We can compute the cohomology groups on  $\mathbb{P}(V)$  by applying the Bott theorem 2.0.1 again. For the calculation we evaluate irreducible decompositions, for example,

$$T(-1)^{\wedge 2} \otimes \Omega(1) \simeq \Sigma^{(0,0,0,-1)}\Omega(1) \oplus \Sigma^{(1,0,-1,-1)}\Omega(1)$$

by the Littlewood-Richardson rule. In this way, we finally obtain the following non-vanishing cohomology groups:

$$\begin{array}{ccccc} H^0(\mathcal{Y}_2, \mathcal{C}_{11}), & H^0(\mathcal{Y}_2, \mathcal{C}_{12}), & H^0(\mathcal{Y}_2, \mathcal{C}_{13}) & \mathbb{C}, & V, & \wedge^2 V \\ & H^0(\mathcal{Y}_2, \mathcal{C}_{22}), & H^0(\mathcal{Y}_2, \mathcal{C}_{23}) & \simeq & \mathbb{C}, & V \\ & & H^0(\mathcal{Y}_2, \mathcal{C}_{33}) & & & \mathbb{C} \end{array}$$

and  $H^0(\mathcal{Y}_2, \mathcal{C}_{44}) \simeq \mathbb{C}$ .

Now let us turn our attention to the cases  $H^\bullet(\mathcal{Y}_2, \mathcal{C}_{i4}(1))$  ( $1 \leq i \leq 3$ ) for which the isomorphism (8.10) does not apply.

$$(1) \ H^\bullet(\widetilde{\mathcal{Y}}, \widetilde{\mathcal{C}}_{14}) \simeq H^\bullet(\mathcal{Y}_2, \mathcal{C}_{14}(1)) \simeq H^0(\mathcal{Y}_2, \mathcal{C}_{14}(1)) \simeq S^2V.$$

For  $\mathcal{C}_{14}(1) = \rho_{\mathcal{Y}_2}^* \overline{\mathcal{C}}_{14}(1)$ , we have

$$\mathcal{C}_{14}(1) = \rho_{\mathcal{Y}_2}^*(\mathcal{S}(L_{\mathcal{Y}_3}) \otimes \mathcal{Q}^*)(M_{\mathcal{Y}_2}) \simeq \rho_{\mathcal{Y}_2}^*(\mathcal{S} \otimes \mathcal{Q}^*(\det \mathcal{Q}))(-F_\rho)$$

by Proposition 6.4.5. Consider the exact sequence

$$\begin{aligned} 0 \rightarrow \rho_{\mathcal{Y}_2}^*(\mathcal{S} \otimes \mathcal{Q}^*(\det \mathcal{Q}))(-F_\rho) &\rightarrow \rho_{\mathcal{Y}_2}^*(\mathcal{S} \otimes \mathcal{Q}^*(\det \mathcal{Q})) \\ &\rightarrow (\rho_{\mathcal{Y}_2}|_{F_\rho})^*(\mathcal{S} \otimes \mathcal{Q}^*(\det \mathcal{Q})|_{\mathcal{P}_\rho}) \rightarrow 0. \end{aligned}$$

We evaluate the cohomology of the middle term by  $H^\bullet(\mathcal{Y}_3, \mathcal{S} \otimes \mathcal{Q}^*(\det \mathcal{Q})) \simeq \oplus_i H^{\bullet-i}(\mathbb{P}(V), R^i_{\pi_{\mathcal{Y}_3}*}(\mathcal{S} \otimes \mathcal{Q}^*(\det \mathcal{Q})))$ . By the Bott theorem for  $\pi_{\mathcal{Y}_3}: \mathcal{Y}_3 \rightarrow \mathbb{P}(V)$ , it is easy to see that the only non-trivial term comes from  $\pi_{\mathcal{Y}_3*}(\mathcal{S} \otimes \mathcal{Q}^*(\det \mathcal{Q})) \simeq \Sigma^{(1,1,0,0,-1)}T(-1)^{\wedge 2}$ . To use the Bott theorem again for the cohomology over  $\mathbb{P}(V)$ , we note the following plethysm of the Schur functors:

$$\Sigma^{(1,1,0,0,-1)}T(-1)^{\wedge 2} \simeq \Sigma^{(2,0,0,0)}T(-1) \oplus \Sigma^{(1,1,1,-1)}T(-1) \oplus \Sigma^{(2,1,0,-1)}T(-1).$$

Then the only non-vanishing result comes from the first summand, and we finally obtain  $H^\bullet(\mathcal{Y}_3, \mathcal{S} \otimes \mathcal{Q}^*(\det \mathcal{Q})) = H^0(\mathbb{P}(V), \Sigma^{(2,0,0,0)}T(-1)) \simeq S^2V$ .

Now, let us note that  $H^\bullet(F_\rho, (\rho_{\mathcal{Y}_2}|_{F_\rho})^*(\mathcal{S} \otimes \mathcal{Q}^*(\det \mathcal{Q})|_{\mathcal{P}_\rho})) \simeq H^\bullet(\mathcal{P}_\rho, \mathcal{S} \otimes \mathcal{Q}^*(\det \mathcal{Q})|_{\mathcal{P}_\rho})$ . By Propositions 6.3.2 and 6.3.3, we have  $\mathcal{S} \otimes \mathcal{Q}^*(\det \mathcal{Q})|_{\mathcal{P}_\rho} \simeq \mathcal{R}_\rho \otimes \mathcal{R}_\rho^*(2 \det \mathcal{R}_\rho - L_{\mathcal{P}_\rho})$ . In a similar way to above by considering the fibration  $\mathcal{P}_\rho \rightarrow \mathbb{P}(V)$ , we see that

$$H^\bullet(\mathcal{P}_\rho, \mathcal{R}_\rho \otimes \mathcal{R}_\rho^*(2 \det \mathcal{R}_\rho - L_{\mathcal{P}_\rho})) \simeq H^\bullet(\mathcal{P}_\rho, \mathcal{O}_{\mathcal{P}_\rho}(2 \det \mathcal{R}_\rho - L_{\mathcal{P}_\rho})).$$

Further, the r.h.s. vanish except possibly  $H^0(\mathcal{P}_\rho, \mathcal{O}_{\mathcal{P}_\rho}(2 \det \mathcal{R}_\rho - L_{\mathcal{P}_\rho}))$  and this is isomorphic to  $H^0(\mathbb{P}(V), S^2T(-1) \otimes \mathcal{O}(-1))$ . This vanishes by the Bott theorem 2.0.1 on  $\mathbb{P}(V)$ .

This completes our calculation of  $H^\bullet(\widetilde{\mathcal{V}}, \widetilde{\mathcal{C}}_{14})$ .

$$(2) \ H^\bullet(\widetilde{\mathcal{V}}, \widetilde{\mathcal{C}}_{34}) \simeq H^\bullet(\mathcal{Y}_2, \mathcal{C}_{34}(1)) \simeq 0.$$

For  $\mathcal{C}_{34}(1) = \rho_{\mathcal{Y}_2}^* \widetilde{\mathcal{C}}_{34}(1)$ , we have

$$\mathcal{C}_{34}(1) = \rho_{\mathcal{Y}_2}^*(\mathcal{Q}^*)(M_{\mathcal{Y}_2}) \simeq \rho_{\mathcal{Y}_2}^*(\mathcal{Q}^*(\det \mathcal{Q} - L_{\mathcal{Y}_3}))(-F_\rho).$$

by Proposition 6.4.5. Since the following calculations proceeds exactly in the same ways as above, we only sketch them.

First, we have  $H^\bullet(\mathcal{Y}_2, \rho_{\mathcal{Y}_2}^*(\mathcal{Q}^*(\det \mathcal{Q} - L_{\mathcal{Y}_3}))) \simeq H^\bullet(\mathcal{Y}_3, \mathcal{Q}^*(\det \mathcal{Q} - L_{\mathcal{Y}_3}))$ , and then evaluate this by the Bott theorem to be  $H^\bullet(\mathbb{P}(V), \Sigma^{(1,1,0,0,0,0)}T(-1)^{\wedge 2} \otimes \mathcal{O}(-1))$ . We use the plethysm  $\Sigma^{(1,1,0,0,0,0)}T(-1)^{\wedge 2} \simeq \Sigma^{(2,1,1,0)}T(-1)$ . Then we evaluate  $H^\bullet(\mathbb{P}(V), \Sigma^{(1,1,0,0,0,0)}T(-1)^{\wedge 2} \otimes \mathcal{O}(-1)) \simeq H^\bullet(\mathbb{P}(V), \Sigma^{1,0,0,-1}T(-1)) = 0$  by the Bott theorem.

Second, we note the isomorphism

$$H^\bullet(F_\rho, (\rho_{\mathcal{Y}_2}|_{F_\rho})^*(\mathcal{Q}^*(\det \mathcal{Q} - L_{\mathcal{Y}_3})|_{\mathcal{P}_\rho})) \simeq H^\bullet(\mathcal{P}_\rho, \mathcal{Q}^*(\det \mathcal{Q} - L_{\mathcal{Y}_3})|_{\mathcal{P}_\rho}).$$

By Proposition 6.3.3, we have now  $\mathcal{Q}^*(\det \mathcal{Q} - L_{\mathcal{Y}_3})|_{\mathcal{P}_\rho} \simeq \mathcal{R}_\rho(H_{\mathbb{P}(T(-1))})$ . Then we see that  $H^\bullet(\mathcal{P}_\rho, \mathcal{Q}^*(\det \mathcal{Q} - L_{\mathcal{Y}_3})|_{\mathcal{P}_\rho}) \simeq H^\bullet(\mathbb{P}(V), \Sigma^{(1,0,0,-1)}\Omega(1))$ , which all vanish.

This completes our calculation of  $H^\bullet(\widetilde{\mathcal{V}}, \widetilde{\mathcal{C}}_{34})$ .

$$(3) \ H^\bullet(\widetilde{\mathcal{V}}, \widetilde{\mathcal{C}}_{24}) \simeq H^\bullet(\mathcal{Y}_2, \mathcal{C}_{24}(1)) \simeq H^0(\mathcal{Y}_2, \mathcal{C}_{24}(1)) \simeq V.$$

Finally, for  $\mathcal{C}_{24}(1) = \mathcal{T} \otimes \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*(M_{\mathcal{Y}_2})$ , we consider the following exact sequence which we derive from (6.19):

$$\begin{aligned} 0 \rightarrow \pi_{\mathcal{Y}_2}^* T(-1) \otimes \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*(M_{\mathcal{Y}_2}) &\rightarrow \mathcal{T} \otimes \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*(M_{\mathcal{Y}_2}) \\ &\rightarrow (\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(-1) \otimes \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*(M_{\mathcal{Y}_2} + F_\rho)|_{F_\rho} \rightarrow 0. \end{aligned}$$

Then for our purpose it suffices to compute  $H^\bullet(\mathcal{Y}_2, \pi_{\mathcal{Y}_2}^* T(-1) \otimes \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*(M_{\mathcal{Y}_2}))$  and also  $H^\bullet(F_\rho, (\rho_{\mathcal{Y}_2}|_{F_\rho})^* \mathcal{O}_{\mathbb{P}(T(-1))}(-1) \otimes \rho_{\mathcal{Y}_2}^* \mathcal{Q}^*(M_{\mathcal{Y}_2} + F_\rho)|_{F_\rho})$ . We can compute the former in a similar way to the above two cases, and we see that they all vanish. Using (1) and (2) of Proposition 6.3.3, we see that the latter cohomologies are isomorphic to  $H^\bullet(\mathcal{P}_\rho, \mathcal{R}_\rho)$ . Now, from the defining exact sequence (6.8) of  $\mathcal{R}_\rho$ , we obtain  $H^\bullet(\mathcal{P}_\rho, \mathcal{R}_\rho) \simeq H^\bullet(\mathbb{P}(V), T(-1))$ , which vanish except  $H^0(\mathbb{P}(V), T(-1)) \simeq V$ .

This completes our calculation of  $H^\bullet(\widetilde{\mathcal{V}}, \widetilde{\mathcal{C}}_{24})$ .

Now our calculations of the cohomologies (8.10) and the cases (1)-(3) above complete our proof of Theorem 8.1.1.

APPENDIX A. THE “DOUBLE SPIN” COORDINATES OF  $G(3, 6)$ 

In this appendix, we set  $V_4 = \mathbb{C}^4$  with the standard basis. We can write the irreducible decomposition (6.6) as

$$\wedge^3(\wedge^2 V_4) = \Sigma^{(3,1,1,1)} V_4 \oplus \Sigma^{(2,2,2,0)} V_4 \simeq S^2 V_4 \oplus S^2 V_4^*,$$

where  $\Sigma^\beta$  is the Schur functor. We define the projective space  $\mathbb{P}(\wedge^3(\wedge^2 V_4)) = \mathbb{P}(S^2 V_4 \oplus S^2 V_4^*)$ . The homogeneous coordinate of  $\mathbb{P}(S^2 V_4 \oplus S^2 V_4^*)$  is naturally introduced by  $[v_{ij}, w^{kl}]$ , where  $v_{ij}$  and  $w^{kl}$  are entries of  $4 \times 4$  symmetric matrices. Let  $\mathcal{I} = \{\{i, j\} \mid 1 \leq i < j \leq 4\}$  the index set to write the standard basis of  $\wedge^2 V_4$ , then the homogeneous coordinate of  $\mathbb{P}(\wedge^3(\wedge^2 V_4))$  is naturally given by the  $[p_{IJK}]$  where  $p_{IJK}$  is totally anti-symmetric for the indices  $I, J, K \in \mathcal{I}$ . These two coordinates are related by the above irreducible decomposition. Focusing the different symmetry properties of the Schur functors, it is rather straightforward to decompose  $p_{IJK}$  into the two components. They are given by

$$(A.1) \quad \begin{aligned} v = (v_{ij}) &= \begin{pmatrix} 2p_{124} & p_{134} + p_{125} & p_{234} + p_{126} & p_{146} - p_{245} \\ & 2p_{135} & p_{235} + p_{136} & p_{156} - p_{345} \\ & & 2p_{236} & p_{256} - p_{346} \\ & & & 2p_{456} \end{pmatrix}, \\ w = (w_{kl}) &= \begin{pmatrix} 2p_{356} & -p_{346} - p_{256} & p_{345} + p_{156} & p_{235} - p_{136} \\ & 2p_{246} & -p_{245} - p_{146} & p_{126} - p_{234} \\ & & 2p_{145} & p_{134} - p_{125} \\ & & & 2p_{123} \end{pmatrix}, \end{aligned}$$

where we ordered the index set  $\mathcal{I}$  as  $\{\mathbf{1}, \mathbf{2}, \dots, \mathbf{6}\} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\}$ . Inverting the relations (A.1), we can write the Plücker relations among  $p_{IJK}$  in terms of the entries of  $v$  and  $w$ . After some algebra, we find:

**Proposition A.1** *The Plücker ideal  $I_G$  of  $G(3, 6) \subset \mathbb{P}(\wedge^3(\wedge^2 V_4))$  is generated by*

$$(A.2) \quad \begin{aligned} &|v_{IJ}| - \epsilon_{I\tilde{I}} \epsilon_{J\tilde{J}} |w_{\tilde{I}\tilde{J}}| \quad (I, J \in \mathcal{I}), \\ &(v.w)_{ij}, \quad (v.w)_{ii} - (v.w)_{jj} \quad (i \neq j, 1 \leq i, j \leq 4), \end{aligned}$$

where  $\tilde{I}$  represents the complement of  $I$ , i.e.,  $x \in \mathcal{I}$  such that  $x \cup I = \{1, 2, 3, 4\}$  and similarly for  $\tilde{J}$ .  $|v_{IJ}|$  and  $|w_{IJ}|$  represent the  $2 \times 2$  minors of  $v$  and  $w$ , respectively, with the rows and columns specified by  $I$  and  $J$ .  $\epsilon_{I\tilde{I}}$  is the signature of the permutation of the ‘ordered’ union  $I \cup \tilde{I}$ .  $(v.w)_{ij}$  is the  $ij$ -entry of the matrix multiplication  $v.w$ .

For all  $[v, w] \in V(I_G) \simeq G(3, 6)$ , we show the following relations (I.1)-(I.5):

$$(I.1) \quad \det v = \det w.$$

By the Laplace expansion of the determinant of  $4 \times 4$  matrix  $v$ , we have  $\det v = \sum_{J \in \mathcal{I}} \epsilon_{J\tilde{J}} |v_{IJ}| |v_{\tilde{I}\tilde{J}}|$ . Then, using the first relations of (A.2), we obtain the equality.

$$(I.2) \quad v.w = \pm \sqrt{\det w} \, id_4, \text{ where } id_4 \text{ is the } 4 \times 4 \text{ identity matrix.}$$

Note that the second line of (A.2) may be written in a matrix form  $v.w = d \, id_4$  with  $d = (v.w)_{11} = \dots = (v.w)_{44}$ . Then, by (I.1), we have  $\det v \cdot w = (\det w)^2 = d^4$  and hence  $d^4 - (\det w)^2 = (d^2 - \det w)(d^2 + \det w) = 0$ . We consider a special case;  $v = a \, id_4$ ,  $w = a \, id_4$ . Then  $d = (v.w)_{11} = a^2$ . Therefore  $d^2 = a^4 = \det w$  must hold for all since  $V(I_G) \simeq G(3, 6)$  is irreducible. Hence  $d = \pm \sqrt{\det w}$  as claimed.

(I.3)  $\operatorname{rk} w \neq 3$  and also  $\operatorname{rk} v \neq 3$ .

Assume  $\operatorname{rk} w = 3$ , then from (I.2) we have  $v.w = 0$ , which implies  $\operatorname{rk} v \leq 1$ . However, this contradicts the first relations of (A.2). Hence  $\operatorname{rk} w \neq 3$ . By symmetry, we also have  $\operatorname{rk} v \neq 3$ .

(I.4)  $\operatorname{rk} w = 2 \Leftrightarrow \operatorname{rk} v = 2$ .

When  $\operatorname{rk} w = 2$ , we see  $\operatorname{rk} v \geq 2$  by the first relations of (A.2). From (I.1) and (I.3), we must have  $\operatorname{rk} v = 2$ . The converse follows in the same way.

(I.5)  $\operatorname{rk} w \leq 1 \Leftrightarrow \operatorname{rk} v \leq 1$ .

This is immediate from the the first relations of (A.2).

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